ON THE STRENGTH OF WEAK COMPACTNESS

ALEXANDER P. KREUZER

Abstract. We study the logical and computational strength of weak compactness in the separable Hilbert space ℓ_2.

Let weak-BW be the statement the every bounded sequence in ℓ_2 has a weak cluster point. It is known that weak-BW is equivalent to ACA_0 over RCA_0 and thus that it is equivalent to (nested uses of) the usual Bolzano-Weierstraß principle BW.

We show that weak-BW is instance-wise equivalent to Π^0_2-CA. This means that for each Π^0_2 sentence $\mathsf{A}(n)$ there is a sequence $(x_i)_{i\in\mathbb{N}}$ in ℓ_2, such that one can define the comprehension function for $\mathsf{A}(n)$ recursively in a cluster point of $(x_i)_i$. As a consequence we obtain that the degrees $d \geq_T 0''$ are exactly the degrees that contain a weak cluster point of any computable, bounded sequence in ℓ_2. Since a cluster point of any sequence in the unit interval $[0, 1]$ can be computed in a degree low over $0'$ (see [Kre11]), this also shows that instances of weak-BW are strictly stronger than instances of BW.

We also comment on the strength of weak-BW in the context of abstract Hilbert spaces in the sense of Kohlenbach and show that his construction of a solution for the functional interpretation of weak compactness is optimal, cf. [Koh].

We investigate the computational and logical strength of weak sequential compactness in the separable Hilbert space ℓ_2.

The strength of weak compactness has so far only been studied in the context of proof mining where general Hilbert spaces in a more general logical system are considered, see [Koh10, Koh]. It is straightforward to deduce from this analysis that weak compactness for ℓ_2 is equivalent to ACA_0 over RCA_0.

In this paper we refine this result and show that weak compactness on ℓ_2 is instance-wise equivalent to Π^0_2-CA over RCA_0. This means that for each bounded sequence in ℓ_2 one can uniformly compute a function f such that from a comprehension function for $\forall x\, \exists y\, f(x, y, n) = 0$ one can compute a weak cluster point of the sequence and vice versa.

As a consequence we obtain that the degrees $d \geq_T 0''$ are exactly the degrees that compute a weak cluster point for each computable bounded sequence in ℓ_2 and that there is a computable bounded sequence in ℓ_2 such that from any cluster point of this sequence one can compute $0''$.

2010 *Mathematics Subject Classification.* Primary 03F60; Secondary 03D80, 03B30.

Key words and phrases. Bolzano-Weierstraß principle, weak sequential compactness, Turing degree, abstract Hilbert space.

The author is supported by the German Science Foundation (DFG Project KO 1737/5-1).

I am grateful to Ulrich Kohlenbach for useful discussions and suggestions for improving the presentation of the material in this article.

I would like to thank the anonymous referees for corrections and suggestions that have helped to improve the revised version of this article.

In [Kre11] we showed that instances of the Bolzano-Weierstraß principle for the unit interval $[0, 1]$ are equivalent to instances Σ^0_1-WKL, i.e. WKL for 0/1-trees given by a Σ^0_1-predicate. Thus instances of the Bolzano-Weierstraß principle for weak compactness are strictly stronger than instances of the usual Bolzano-Weierstraß principle.

This paper is organized as follows: first the Hilbert space ℓ_2 is defined. This definition follows [Sim09, AS06]. Then the actual results are proven (Theorems 9 and 13) and we show that the result can also be formulated for abstract Hilbert spaces, in the sense of Kohlenbach [Koh08] (Theorem 11). As a corollary of this we obtain that Kohlenbach's analysis of the weak compactness functional Ω^* in [Koh] is optimal (Corollary 12). At the end, we reformulate our result in terms of the Weihrauch lattice (Remark 15).

Definition 1 (vector space, [Sim09, II.10])**.** A *countable vector space A over a countable field K* consists of a set $|A| \subseteq \mathbb{N}$ with operations $+\colon |A| \times |A| \to |A|$ and $\cdot\colon |K| \times |A| \to |A|$ and a distinguished element $0 \in |A|$ such that $(|A|, +, \cdot, 0)$ satisfies the usual axioms for a vector space over K.

Definition 2 (Hilbert space, [AS06, Definition 9.3])**.** A *(real) separable Hilbert space H* consists of a countable vector space A over $\mathbb{Q}$ together with a function $\langle\cdot|\cdot\rangle\colon A \times A \to \mathbb{R}$ satisfying

(1) $\langle x|x\rangle \geq 0$,
(2) $\langle x|y\rangle = \langle y|x\rangle$,
(3) $\langle ax + by|z\rangle = a\langle x|z\rangle + b\langle y|z\rangle$,

for all $x, y, z \in A$ and $a, b \in \mathbb{Q}$.

The inner product $\langle\cdot|\cdot\rangle$ on H induces a pseudonorm $\|x\| := \sqrt{\langle x|x\rangle}$. We think of the Hilbert space H as the completion of A under the pseudometric $d(x, y) = \|x - y\|$. Thus an element of H consists of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, such that $d(x_n, x_m) < 2^{-n}$ for all $m > n$. The inner product and the pseudonorm are continuously extended to the whole space H.

A Hilbert space is finite dimensional if it is spanned by finitely many vectors. If this is not the case we say that it is infinite dimensional.

Avigad, Simic showed in [AS06, Theorem 10.9] that RCA_0 proves that every Hilbert space H in the sense of Definition 2 has an orthonormal basis. Since each such Hilbert space is separable this basis is at most countable.

As consequence of this each two infinite dimensional (separable) Hilbert spaces are isomorphic over RCA_0, see [AS06, Corollary 10.11]. Thus we many restrict our attention to ℓ_2, as given by the following definition.

Definition 3 (ℓ_2, [Sim09, II.10.2])**.** Let $A = (|A|, +, \cdot, 0)$ be the following vector space over $\mathbb{Q}$, where $|A|$ is the set of all finite sequences of rational numbers $\langle r_0, \dots, r_m\rangle$, such that either $m = 0$ or $r_m \neq 0$. Addition is defined by putting $\langle r_0, \dots, r_m\rangle + \langle s_0, \dots, s_n\rangle = \langle r_0 + s_0, \dots, r_k + s_k\rangle$ where $r_i = 0$, $s_j = 0$ for $i > m$, $j > n$ and $k = \max\{i \mid i = 0 \vee r_i + s_i \neq 0\}$. For scalar multiplication put $q \cdot \langle r_0, \dots, r_m\rangle = \langle 0\rangle$ if $q = 0$ and $\langle q \cdot r_0, \dots, q \cdot r_m\rangle$ otherwise.

The space ℓ_2 is defined to be the Hilbert space consisting of A with the inner product

$$\big\langle\langle r_0, \dots, r_m\rangle\big|\langle s_0, \dots, s_n\rangle\big\rangle = \sum_{i=0}^{\max(n,m)} r_i s_i.$$

The canonical orthonormal basis $(e_n)_n$ of ℓ_2 is given by

$$e_n = \langle\underbrace{0,\dots,0}_{n \text{ times}},1\rangle.$$

Definition 4 (bounded linear operator, [AS06, Definition 9.4])**.** A *bounded linear operator* from a Hilbert space $H = (A, \langle\cdot|\cdot\rangle)$ to a Hilbert space $H' = (A', \langle\cdot|\cdot\rangle)$ is a function $F\colon |A| \to A'$, such that

- F is linear, i.e. $F(q_1x_1 + q_2x_2) = q_1F(x_1) + q_2F(x_2)$ for all $q_1, q_2 \in \mathbb{Q}$ and $x_1, x_2 \in |A|$ and
- the norm of F is bounded, i.e. there exists an $m \in \mathbb{R}$ with $\|F(x)\| \leq m\|x\|$ for all $x \in |A|$.

Then, F is continuously extended to the whole space A.

Definition 5 (projection)**.** Let M be a closed linear subspace of a Hilbert space H. A point $y \in M$ is called the *projection* of $x \in H$ if $x - y$ is orthogonal to (each element of) M, i.e. $\forall z \in M\ \langle z|x - y\rangle = 0$.

A bounded linear operator P_M on H that maps each point of H to its projection on M is called the *projection function for* M.

Usually projections are defined differently, see e.g. [AS06, Definition 12.1]. Avigad, Simic showed that this definition is equivalent over RCA_0 to the usual definition, see [AS06, Lemma 12.2].

We immediately obtain the following lemma:

Lemma 6. *Let $N \subset \mathbb{N}$ and M be the subspace of ℓ_2 that is spanned by $\{e_n \mid n \in N\}$. Then RCA_0 proves that the projection P_M of ℓ_2 onto the space M exists.*

Proof. The projection of an element $\langle r_0, \dots, r_m\rangle$ of the space $|A|$ is given by the vector $\langle r'_0, \dots, r'_{m'}\rangle$, where $r'_i = r_i$ if $n \in N$ and $r'_i = 0$ if $n \notin N$ and $m' = \max\{i \leq m \mid r'_i \neq 0 \vee i = 0\}$.

It is easy to show that P_M is linear and that it is bounded by 1. Thus it defines a bounded linear operator in the sense of Definition 4. □

Definition 7 (weak convergence)**.** We say that a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of a Hilbert space H *converges weakly* to a point x if

$$\forall y \in H\ \lim_{i\to\infty} \langle y|x_i\rangle = \langle y|x\rangle. \tag{1}$$

The *Bolzano-Weierstraß principle for weak convergence* is defined to be the statement that for every bounded sequence $(x_i)_{i\in\mathbb{N}}$ of elements of H there exists a point x such that a subsequence of $(x_i)_i$ converges weakly to x. This principle is abbreviated by $\mathsf{weak\text{-}BW}$. The restriction of this principle to a fixed sequence $(x_i)_{i\in\mathbb{N}}$ is denoted by $\mathsf{weak\text{-}BW}((x_i)_i)$. If H has an orthonormal basis it is sufficient to have (1) only for all y in the basis.

Lemma 8. *The system RCA_0 proves that projections are weakly continuous in the sense that if x is the weak limit of a sequence $(x_i)_{i\in\mathbb{N}}$, then Px is the weak limit of $(Px_i)_{i\in\mathbb{N}}$ for any projection P.*

Proof. Follows from the definition of the projection and the continuity of $\langle\cdot|\cdot\rangle$. □

We will denote by $\Pi^0_2\text{-}\mathsf{CA}(h)$ the instance of Π^0_2-comprehension given by the formula $A(n) \equiv \forall x\, \exists y\, h(x, y, n) = 0$, i.e. the statement

$$\exists g\, \forall n\ (g(n) = 0 \leftrightarrow \forall x\, \exists y\, h(x, y, n) = 0)\,.$$

Theorem 9. *Each instance of* Π^0_2-CA *given by the formula* $\forall x\, \exists y\, h(x,y,n)=0$ *is uniformly implied by an instance of* $\mathsf{weak\text{-}BW}$*. More precisely, there exists a type* 2 *program* F*, such that*

$$\mathsf{RCA}_0 \vdash \forall h\, \left(\mathsf{weak\text{-}BW}(F(h)) \rightarrow \Pi^0_2\text{-}\mathsf{CA}(h)\right).$$

(In RCA_0 *the program* $F(h)$ *can be formulated as an oracle Turing machine* $\{\mathsf{e}\}^h$ *for a suitable* $\mathsf{e} \in \mathbb{N}$*.)*

Proof. Fix an h and define

$$f(n,i) := \max\{x \le i \mid \forall x' < x\, \exists y < i\, (h(x',y,n)=0)\}.$$

It is clear that $\lambda i.f(n,i)$ is increasing for each n.
Claim 1.

$$\mathsf{A}(n) \quad \text{iff} \quad \lambda i.f(n,i) \text{ is unbounded, i.e. } \forall k\, \exists i\, (f(n,i) > k).$$

Proof of Claim 1.

- The right to left direction follows immediately from the definition of f.
- For the left to right direction fix an n. We will show that the negation of the right side implies the negation of the left side.
  Hence assume that $\lambda i.f(n,i)$ is bounded by k, i.e.

$$\forall i\, (f(n,i) \le k). \tag{2}$$

  By Σ^0_1-induction we may assume that k is minimal and thus

$$\exists i\, (f(n,i) = k).$$

  From the definition of f we obtain

$$\forall x < k\, \exists y\, (h(x,y,n)=0).$$

  Together with (2) we obtain that

$$\forall y\, (h(k,y,n) \ne 0)$$

  and hence $\neg\mathsf{A}(n)$.

This proves the claim.
Let

$$y_{n,i} := e_{\langle n, f(n,i)\rangle}.$$

The sequence $(y_{n,i})_{i\in\mathbb{N}}$ is obviously bounded by 1 and hence possesses for each n a weak cluster point y_n.
Claim 2.

- $\|y_n\| =_{\mathbb{R}} 0$, if $\mathsf{A}(n)$ and
- $\|y_n\| =_{\mathbb{R}} 1$, if $\neg\mathsf{A}(n)$.

Proof of Claim 2.

- If $\mathsf{A}(n)$ is true, then $\lambda i.f(n,i)$ is unbounded and hence $\langle e_j | y_{n,i}\rangle$ becomes 0 as i increases. Therefore $y_{n,i}$ converges weakly to 0.
- If $\mathsf{A}(n)$ is false, then $\lambda i.f(n,i)$ is bounded. By Σ^0_1-induction we obtain a smallest upper bound k and since $\lambda i.f(n,i)$ is increasing we obtain that $\lim_{i\to\infty} f(n,i) = k$. As a consequence we obtain that $y_{n,i}$ eventually becomes $e_{\langle n,k\rangle}$ and hence that $y_n = e_{\langle n,k\rangle}$ and $\|y_n\| =_{\mathbb{R}} 1$.

This proves the claim.

We parallelize this process to obtain the comprehension function for $\mathsf{A}(n)$. For this let

$$x_i := \sum_{n=0}^{i} 2^{-\frac{n+1}{2}} y_{n,i}.$$

Since the $y_{n,i}$ are orthogonal for different n, we obtain by Pythagoras that

$$\|x_i\|^2 = \sum_{n=0}^{i} 2^{-(n+1)} \|y_{n,i}\|^2 \leq 1$$

and thus that (x_i) is bounded.

It is also clear that there exists an F such that $x_i = F(h, i)$.

By $\mathsf{weak\text{-}BW}(F(h))$ there exists a weak cluster point x of (x_i). Let now M_n be the closed linear space spanned by $\{e_{\langle n,k\rangle} \mid k \in \mathbb{N}\}$. By definition the subspaces M_n are disjoint (except for the 0 vector) for different n, and $y_{n,i} \in M_n$ for all i, n.

By Lemma 6 the projections P_{M_n} onto the spaces M_n exist. For this projections we have

$$P_{M_n}(x_i) = 2^{-\frac{n+1}{2}} y_{n,i} \quad \text{for } n \geq i.$$

Since P_{M_n} is weakly continuous, see Lemma 8, we get

$$P_{M_n}(x) = 2^{-\frac{n+1}{2}} y_n.$$

Now Claim 2 yields that $\|P_{M_n}(x)\| =_{\mathbb{R}} 0$ if $\mathsf{A}(n)$ and $\|P_{M_n}(x)\| =_{\mathbb{R}} 2^{-\frac{n+1}{2}}$ if $\neg\mathsf{A}(n)$. Hence the function

$$g(n) := \begin{cases} 0 & \text{if } \|P_{M_n}(x)\|(n+1) <_{\mathbb{Q}} 2^{-\frac{n+1}{2}}, \\ 1 & \text{otherwise,} \end{cases}$$

where $\|P_{M_n}(x)\|(n+1)$ is a $2^{-(n+1)}$ good rational approximation of $\|P_{M_n}(x)\|$, provides a comprehension function and solves the theorem. □

As an immediate consequence we obtain the following corollary:

Corollary 10. *There is a sequence $(x_i)_i$ of elements in ℓ_2 such that from a cluster point x of this sequence one can compute the second Turing jump $0''$.*

Proof. Take for $\forall x\, \exists y\, h(x, y, n) = 0$ in Theorem 9 the Π^0_2 statement that the Turing machine $\{n\}^{0'}(n)$ does not halt. □

Kohlenbach studies weak compactness in the context of arbitrary abstract Hilbert spaces, see [Koh08, Koh10]. By abstract Hilbert space we mean that the Hilbert space is added as a new type to the system together with the Hilbert space axioms and that the space is not coded as sequences of numbers. In this way one can analyze Hilbert spaces without referring to a concrete space like ℓ_2 and one does not automatically obtain a separable Hilbert space but can analyze general Hilbert spaces.

For this, we will work in the system $\widehat{\mathsf{PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,0}$. This is roughly the extension of RCA_0 to finite types. ($\mathsf{QF\text{-}AC}^{0,0}$ denotes quantifier-free choice of numbers over numbers and is equivalent to Δ^0_1-CA.) The terms of $\widehat{\mathsf{PA}}^\omega\!\restriction$ correspond to the extension of Kleene's primitive recursive functionals to mixed types and are called T_0. For a definition of $\widehat{\mathsf{PA}}^\omega\!\restriction$ see for instance [Koh08, Chapter 3] and note that the system $\widehat{\mathsf{PA}}^\omega\!\restriction$ is there called $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\restriction$. By Π^0_1-CP we denote the Π^0_1-bounded collection principle. (In first order context Π^0_1-CP is sometimes denoted by $B\Pi_1$.) We do not introduce the notation for abstract Hilbert spaces here but refer the reader to [Koh08, Chapter 17]. We show now that the statement of Theorem 9 is also applicable in this context:

Theorem 11. *Let $\widehat{\mathsf{PA}}^\omega\!\restriction[X,\langle\cdot|\cdot\rangle]$ be the extension of $\widehat{\mathsf{PA}}^\omega\!\restriction$ by the abstract Hilbert space X with the inner product $\langle\cdot|\cdot\rangle$ and let $\mathsf{weak}\text{-}\mathsf{BW}_X$ denote the Bolzano-Weierstraß principle for weak compactness in X.*

Then there is a closed term $F \in T_0$, such that

$$\widehat{\mathsf{PA}}^\omega\!\restriction[X,\langle\cdot|\cdot\rangle] + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} + \Pi^0_1\text{-}\mathsf{CP} \vdash \forall h\, \forall (e_i)_{i\in\mathbb{N}} \big(\forall i,j\, \langle e_i|e_j\rangle = \delta_{ij} \\ \to \big(\mathsf{weak}\text{-}\mathsf{BW}_X(F((e_i)_i,h)) \to \Pi^0_2\text{-}\mathsf{CA}(h)\big)\big),$$

where δ_{ij} is a shorthand for $\begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$

In other words, if X is provably infinite dimensional and $(e_i)_i$ is a witness for that, then Theorem 9 also holds with ℓ_2 replaced by X.

Proof. The only step in the proof of Theorem 9 that does not formalize in the system $\widehat{\mathsf{PA}}^\omega\!\restriction[X,\langle\cdot|\cdot\rangle]$ is projection of x onto M_n, i.e. Lemma 6, since this depends on the coding of ℓ_2.

Let x_i, $y_{n,i}$ be as in Theorem 9. We show now how to obtain this projection of x in this system. For this consider

$$\begin{aligned} \|x\|^2 = \langle x|x\rangle &= \lim_{i\to\infty} \langle x|x_i\rangle \\ &= \lim_{i\to\infty} \sum_{n=0}^{i} 2^{-(n+1)} \langle x|y_{n,i}\rangle \\ &\le \lim_{i\to\infty} \sum_{n=0}^{k} 2^{-(n+1)} \langle x|y_{n,i}\rangle + 2^{-k} \qquad \text{for each } k \\ &= \sum_{n=0}^{k} 2^{-(n+1)} \lim_{i\to\infty} \langle x|y_{n,i}\rangle + 2^{-k}. \end{aligned}$$

Now

$$\langle x|y_{n,i}\rangle = \lim_{j\to\infty} \langle x_j|y_{n,i}\rangle = 2^{-(n+1)} \lim_{j\to\infty} \langle y_{n,j}|y_{n,i}\rangle. \tag{3}$$

Thus, by the definition of $y_{n,i}$ the term $\langle x|y_{n,i}\rangle$ is monotone in i and in particular for each n there is an m, such that

$$\lim_{i\to\infty} \langle x|y_{n,i}\rangle = \langle x|y_{n,i'}\rangle \quad \text{for } i' \ge m.$$

By Π^0_1-CP there is now an m which does it for all $n \le k$. Hence, we obtain

$$\forall k\, \exists i\ \|x\|^2 \le \sum_{n=0}^{k} 2^{-(n+1)} \langle x|y_{n,i}\rangle + 2^{-k}.$$

By (3) the term $\langle x|y_{n,i}\rangle$ is either 0 or $2^{-(n+1)}$, hence

$$\forall k\, \exists i\ \|x\|^2 \leq \sum_{n=0}^{k} \langle x|y_{n,i}\rangle^2 + 2^{-k}.$$

Thus, $\sum_{n=0}^{k}\langle x|y_{n,i}\rangle y_{n,i}$ is a $2^{-k/2}$ good approximation of x consisting of finite linear combinations of (e_i). Using an application of $\mathsf{QF\text{-}AC}^{0,0}$ one easily obtains a sequence of approximations converging to x at the rate 2^{-k}. Using this one can obtain $P_{M_n}(x)$ like in Lemma 6. □

Let T_n be the extension of T_0 by iteration for type n objects. The functions contained in T_n are exactly the functions that are provably total with Σ^0_{n+1}-IA.

By applying the functional interpretation to Theorem 11 we obtain the following corollary:

Corollary 12. *Let Ω be a solution of the functional interpretation of* $\mathsf{weak\text{-}BW}_X$ *then for every $n \geq 1$ there are terms in T_n, such that the application of Ω to these terms is (extensionally) equal to a function definable in the T_{n+2} but not in T_{n+1}.*

Proof. Let A be the statement that the function $f_{\omega_{n+1}}$ from the fast growing hierarchy is total. It is well known that the statement A cannot be proven in Σ^0_{n+2}-IA but can be proven using a suitable instance of Σ^0_{n+3}-IA, see [HP98, II.3.(d)]. Thus a solution of the functional interpretation of A cannot be found in T_{n+1} but can be found in T_{n+2}.

Let $\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot|\cdot\rangle, (e_i)_{i\in\mathbb{N}}]$ be the extension of $\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot|\cdot\rangle]$ by the constant $(e_i)_i$, which can be majorized by $\lambda i.1$, and the axiom $\forall i, j \in \mathbb{N}\, \langle e_i|e_j\rangle =_{\mathbb{R}} \delta_{ij}$. For this system the metatheorem [Koh08, Theorem 17.69.2], see also [GK08],

- relativized to the fragment $\widehat{\mathsf{PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,0}$ of $\mathcal{A}^\omega$, cf. [Koh08, Section 17.1, p. 382] and
- extended by the constant $(e_i)_i$ and the purely universal axiom for it, cf. [Koh08, Section 17.5]

holds.

By Theorem 11 a suitable instance of $\mathsf{weak\text{-}BW}_X$ can reduce an instance of Σ^0_{n+3}-IA to Σ^0_{n+1}-IA. Thus the system $\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot|\cdot\rangle, (e_i)_{i\in\mathbb{N}}] + \mathsf{QF\text{-}AC}^{0,0} + \Sigma^0_{n+1}$-$\mathsf{IA}$ proves that a suitable instance of $\mathsf{weak\text{-}BW}_X$ implies A. Applying the metatheorem to this statement yields terms in T_n such that an application of these terms to Ω yields a solution of the functional interpretation of A.

This proves the corollary. □

This shows that Kohlenbach's analysis of Ω^* (a majorant of a solution of the functional interpretation of $\mathsf{weak\text{-}BW}_X$) in [Koh] is optimal.

This analysis and actually even his proof of weak compactness for abstract Hilbert spaces [Koh10, Theorem 11] shows that only two nested instances of Π^0_1-CA (plus some uses of WKL) are needed to prove an instance of $\mathsf{weak\text{-}BW}_X$. Thus, the lower bound on the strength of instances of $\mathsf{weak\text{-}BW}_X$ from Theorems 9 and 11 is strict in the sense that there is a instance of Π^0_3-CA which is not implied by any instance of $\mathsf{weak\text{-}BW}_X$.

We now give a reversal for the special case of ℓ_2 and analyze the exact computational content:

Theorem 13. *Each instance of* weak-BW *given by a bounded sequence* $(x_i)_{i\in\mathbb{N}}$ *in* ℓ_2 *is over* RCA_0 *uniformly provable from a suitable instance of* Π^0_2-CA*. More precisely, there is a program* F *such that*

$$\mathsf{RCA}_0 \vdash \forall (x_i)\ \left(\Pi^0_2\text{-}\mathsf{CA}(F((x_i))) \rightarrow \text{weak-BW}((x_i)_{i\in\mathbb{N}})\right).$$

(Again, in RCA_0 *the program* $F(h)$ *can be coded as an oracle Turing machine* $\{\mathsf{e}\}^h$ *for a suitable* $\mathsf{e}\in\mathbb{N}$*.)*

In particular, each bounded and computable sequences of ℓ_2 *has a weak cluster point computable in* $0''$*.*

Proof. We show that provably in RCA_0 a cluster point of $(x_i)_i$ can be computed in the second Turing jump. The result follows then from the fact that any function computable in the second Turing jump is recursive in a suitable instance of Π^0_2-CA.

We assume that $(x_i)_i$ is bounded by 1.

Note that the Bolzano-Weierstraß theorem for the space $[-1,1]^{\mathbb{N}}$ (with the product metric $d\big((x_i)_i,(y_i)_i\big) = \sum_{i=0}^{\infty}\frac{\min(|x_i-y_i|,1)}{2^{i+1}}$) is instance-wise equivalent to the Bolzano-Weierstraß theorem for $[-1,1]$. This can easily be seen from the fact that the Bolzano-Weierstraß theorem for $[-1,1]$ is instance-wise equivalent to the theorem for the Cantor space $2^{\mathbb{N}}$ and the fact that $2^{\mathbb{N}}$ is isomorphic to $(2^{\mathbb{N}})^{\mathbb{N}}$.

Hence by [Kre11], see also [SK10], one can find a cluster point of the sequence

$$y_i := \big(\langle e_0|x_i\rangle, \langle e_1|x_i\rangle, \dots\big)$$

in $[-1,1]^{\mathbb{N}}$ by computing an infinite path trough a Σ^0_1-tree. Call this cluster point $c=(c_0,c_1,\dots)\in[-1,1]^{\mathbb{N}}$.

Claim. $\sum_{j=0}^{\infty} c_j \le 1$

Proof of claim. Since each y_i is norm bounded by 1 we have that $\sum_{j=0}^{k}(y_i)_j^2 \le 1$. Now for each k and for each ε there is an i such that $|c_j-(y_i)_j|\le\varepsilon$ for $j\le k$ and hence

$$\sum_{j=0}^{k}(c_j)^2 \le \sum_{j=0}^{k}((y_i)_j+\varepsilon)^2 \le 1+3(k+1)\varepsilon.$$

From this the claim follows.

Now one easily checks that the sequence $(z_i)_{i\in\mathbb{N}}$ with $z_i := \langle c_0,\dots,c_i\rangle$ converges in the ℓ_2-norm to a weak cluster point x of $(x_i)_i$. This convergence is monotone in the sense that $\|z_i\|\le\|z_{i+1}\|$. Thus the limit point x can be computed in the Turing jump of $(z_i)_i$.

The point x is provably uniformly computable in the second Turing jump of $(x_i)_i$ because c is computable in a degree provably low over the first Turing jump by the low basis theorem ([JS72]). The proof of the low basis theorem is effective and uniform and it formalizes in RCA_0. Therefore the jump of $(z_i)_i$ and thus x is computable in the second Turing jump and one can find a suitable F. □

Theorems 9 and 13 yield a classification of the computational strength of weak compactness on ℓ_2:

Corollary 14. *For a Turing degree* d *the following are equivalent:*

- $d \ge_T 0''$ *and*
- d *computes a weak cluster point for each computable, bounded sequence in* ℓ_2*.*

As a consequence we obtain that the Bolzano-Weierstraß principle for weak compactness is instance-wise strictly stronger than the Bolzano-Weierstraß principle for the unit interval $[0,1]$, cf. [Kre11].

Remark 15 (Weihrauch lattice). The proofs of the Theorems 9 and 13 can also be used to classify the Bolzano-Weierstraß principle for weak compactness in ℓ_2 in the Weihrauch lattice. We do not introduce the notation for the Weihrauch lattice but refer the reader to [BGM12].

Let $\mathsf{BWT}_{\text{weak-}\ell_2} :\subseteq (\ell_2)^{\mathbb{N}} \rightrightarrows \ell_2$ be the partial multifunction which maps bounded sequences of ℓ_2 to a weak cluster point of that sequence.

The proof of Theorem 9 immediately yields that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \geq_{\mathrm{W}} \widehat{\mathsf{LPO}} \circ \widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \lim^{(2)}.$$

Whereas the proof of Theorem 13 yields that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \leq_{\mathrm{W}} \mathsf{MCT} * \mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}.$$

The function $\mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}$ is used to compute the cluster point $c \in \mathbb{R}^{\mathbb{N}}$, the function MCT is used for the convergence of $(\|z_i\|)_i$. By the same argument as in the proof $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}$. Since all of these multifunctions are cylinders one may also strengthen the reducibility to strong Weihrauch reducibility. Thus

$$\begin{aligned}\mathsf{BWT}_{\text{weak-}\ell_2} &\leq_{\mathrm{sW}} \mathsf{MCT} *_{\mathrm{s}} \mathsf{BWT}_{\mathbb{R}} \\ &\leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathfrak{L}' \\ &\leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathfrak{L}_{1,1} \\ &\equiv_{\mathrm{sW}} \lim \circ \lim.\end{aligned}$$

(For the last equivalence see [BGM12, Corollary 8.8], which is a consequence of an analysis of the low basis theorem in the Weihrauch lattice, see [BdBP12].)

In total we obtain that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \equiv_{\mathrm{sW}} \lim^{(2)}.$$

As consequence we also obtain that $\mathsf{BWT}_{\text{weak-}\ell_2} >_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$.

References

[AS06] Jeremy Avigad and Ksenija Simic, *Fundamental notions of analysis in subsystems of second-order arithmetic*, Ann. Pure Appl. Logic **139** (2006), no. 1-3, 138–184. MR 2206254

[BdBP12] Vasco Brattka, Matthew de Brecht, and Arno Pauly, *Closed choice and a uniform low basis theorem*, Ann. Pure Appl. Logic **163** (2012), no. 8, 986–1008.

[BGM12] Vasco Brattka, Guido Gherardi, and Alberto Marcone, *The Bolzano-Weierstrass theorem is the jump of weak König's lemma*, Ann. Pure Appl. Logic **163** (2012), no. 6, 623–655, Computability in Europe 2010.

[GK08] Philipp Gerhardy and Ulrich Kohlenbach, *General logical metatheorems for functional analysis*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2615–2660. MR 2373327

[HP98] Petr Hájek and Pavel Pudlák, *Metamathematics of first-order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998, Second printing. MR 1748522

[JS72] Carl G. Jockusch, Jr. and Robert I. Soare, Π^0_1 *classes and degrees of theories*, Trans. Amer. Math. Soc. **173** (1972), 33–56. MR 0316227

[Koh] Ulrich Kohlenbach, *Gödel functional interpretation and weak compactness*, Ann. Pure Appl. Logic, published online 24 December 2011, available at `http://dx.doi.org/10.1016/j.apal.2011.12.009`.

[Koh08] ———, *Applied proof theory: Proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer Verlag, 2008. MR 2445721

[Koh10] ______, *On the logical analysis of proofs based on nonseparable Hilbert space theory*, Proofs, Categories and Computations. Essays in Honor of Grigori Mints (S. Feferman and W. Sieg, eds.), College Publications, 2010.

[Kre11] Alexander P. Kreuzer, *The cohesive principle and the Bolzano-Weierstraß principle*, Math. Logic Quart. **57** (2011), no. 3, 292–298. MR 2839129

[Sim09] Stephen G. Simpson, *Subsystems of second order arithmetic*, second ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. MR 2517689

[SK10] Pavol Safarik and Ulrich Kohlenbach, *On the computational content of the Bolzano-Weierstraß principle*, Math. Log. Q. **56** (2010), no. 5, 508–532. MR 2742886

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany

E-mail address: akreuzer@mathematik.tu-darmstadt.de