From Bolzano-Weierstraß to Arzelà-Ascoli

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Bolzano-Weierstraß

A real number is a sequence of rational numbers with Cauchy-rate 2^{-k}.

Definition

(BW):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ has a cluster point.

Equivalently, every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence $(x_{g(n)})$ with Cauchy-rate 2^{-k}, i.e. with

$$\forall k \, \forall n, n' \geq k \, \left(\left| x_{g(n)} - x_{g(n')} \right| < 2^{-k} \right).$$

Definition

(BW_{weak}):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence $(x_{g(n)})$, i.e.

$$\forall k \, \exists m \, \forall n, n' \geq m \, \left(\left| x_{g(n)} - x_{g(n')} \right| < 2^{-k} \right).$$

Status BW

Theorem (Friedman '76)

$\mathrm{RCA}_0 \vdash \mathrm{ACA} \leftrightarrow \mathrm{BW}$.

Theorem (Kohlenbach '98, Kohlenbach, Safarik '10, K. '11)

- *For each computable sequence* (x_n) *there is a* $0'$*-computable* $0/1$*-tree* T*, such that an infinite branch of* T *computes a cluster point, and vice versa.*
- *Over* RCA_0 *the principles* BW *and* WKL *for* Σ^0_1*-trees are* ***instance-wise*** *equivalent.*

Theorem (Brattka, Gherardi, Marcone '12)

$\mathrm{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathrm{WKL}'$.

Status $\mathrm{BW}_{\mathrm{weak}}$

Theorem (K. '11)

For each bounded, computable sequence (x_n) *there is an infinite* $0'$*-computable 0/1-tree* T*, such that a Cauchy-subsequence* $(x_{g(n)})$ *and* $(x_{g(n)})'$ *are computable in an infinite branch of* T*.*

Corollary

$\mathrm{BW}_{\mathrm{weak}}$ *has low*$_2$ *solutions, i.e.* $(x_{g(n)})''$ *is computable in* $0''$*.*

Theorem (Le Roux, Ziegler '08)

There is a computable sequence (x_n) *that has no converging subsequence that is computable in* $0'$*.*

In fact $\mathrm{BW}_{\mathrm{weak}}$ is equivalent to the so called *strong cohesive principle*.

Arzelà-Ascoli

Arzelà-Ascoli theorem

Let $f_n\colon [0,1] \to [0,1]$ be an equicontinuous sequence of functions. Then there exists a subsequence $\left(f_{g(n)}\right)_{n\in\mathbb{N}}$ which converges uniformly.

$f_n\colon [0,1] \to [0,1]$ is called *equicontinuous* if

$$\forall l\,\forall x \in [0,1]\,\exists j\,\forall n\,\forall y \in [0,1]\ \left(|x-y| < 2^{-j} \to |f_n(x) - f_n(y)| < 2^{-l}\right).$$

We assume here that a continuous modulus of equicontinuity $\varphi(x,l)$ exists.

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$f_n\colon [0,1] \to [0,1]$ is called *equicontinuous* if

$$\forall l\, \forall x \in [0,1]\, \exists j\, \forall n\, \forall y \in [0,1]\, \left(|x-y| < 2^{-\varphi(x,l)} \to |f_n(x) - f_n(y)| < 2^{-l}\right).$$

We assume here that a continuous modulus of equicontinuity $\varphi(x,l)$ exists.

Formalization

Let $f_n\colon [0,1] \to [0,1]$ be an equicontinuous sequence of functions.

- AA: There exists subsequence $\left(f_{g(n)}\right)$ which converges at the rate 2^{-k}, i.e.

$$\forall k \, \forall n, n' > \mathrm{k} \;\; \left\| f_{g(n)} - f_{g(n')} \right\|_\infty < 2^{-k}.$$

- $\mathrm{AA}_{\mathrm{weak}}$: There exits a converging subsequence $\left(f_{g(n)}\right)$, i.e.

$$\forall k \, \exists m \, \forall n, n' > m \; \left\| f_{g(n)} - f_{g(n')} \right\|_\infty < 2^{-k}.$$

How hard is it to compute solutions to AA and $\mathrm{AA}_{\mathrm{weak}}$?

The Bolzano-Weierstraß principle is a lower bound:

- Let $(y_n) \subseteq [0, 1]$.
- Define $f_n(x) := y_n$.
  Clearly, (f_n) is equicontinuous.
- If $(f_{g(n)})$ is a uniformly converging subsequence
  then $(y_{g(n)})$ also converges (at the same rate).

Proposition

- *Solutions to* AA *are at least as hard to compute as solutions to* BW*.*
- *Solutions to* $\mathrm{AA}_{\mathrm{weak}}$ *are at least as hard to compute as solutions to* $\mathrm{BW}_{\mathrm{weak}}$*.*

Upper bound on $\mathrm{AA}_{\mathrm{weak}}$

- Let $f_n \colon [0,1] \to [0,1]$ be an equicontinuous sequence of functions.
- We additionally assume that (f_n) is uniformly equicontinuous i.e. the modulus of equicontinuity is independent of x, that is $\varphi(x,l) = \varphi_u(l)$.
- On a $2^{-\varphi_u(l)}$ long interval the functions f_n only vary 2^{-l}.
- We have

$$\|f_n - f_m\|_\infty < 2^{-k}$$

if the following holds

$$|f_n(x) - f_m(x)| < 2^{-k+2}$$

for $x \in \left\{ \frac{i}{2^{-(\varphi_u(k+2)+1)}} \;\middle|\; 0 \le i \le 2^{-(\varphi_u(k+2)+1)} \right\}$.
- Thus, to have uniform convergence of (f_n) it suffices to have convergence of $(f_n(x))$ for all of these x.

Let q_i be an enumeration of $\mathbb{Q} \cap [0,1]$.
Define

$$F \colon f \longmapsto \big(f(q_i)\big)_{i \in \mathbb{N}} \in [0,1]^{\mathbb{N}},$$

where $[0,1]^{\mathbb{N}}$ is the product space with the product metric

$$d((x_i),(y_i)) = \sum_{i=0}^{\infty} 2^{-i} \left| x_i - y_i \right| .$$

For a subsequence $\left(f_{g(n)}\right)$ we then have

$f_{g(n)}$ converges uniformly iff $F(f_{g(n)})$ converges in $[0,1]^{\mathbb{N}}$

Proposition

A uniformly converging subsequence of (f_n) can be computed from a solution of a suitable instance of $\mathrm{BW}_{\mathrm{weak}}$ for the space $[0,1]^{\mathbb{N}}$ under the assumption that φ_u exists.

- $\varphi_u(l)$ exists by a compactness argument, i.e. one can take

$$\varphi_u(l) := \sup_{x \in [0,1]} \varphi(x, l)$$

- $\mathrm{BW}_{\mathrm{weak}}$ for the space $[0,1]^{\mathbb{N}}$ can be reduced to $\mathrm{BW}_{\mathrm{weak}}$ (for the space $[0,1]$) via the homeomorphism

$$[0,1]^{\mathbb{N}} \longrightarrow \left(2^{\mathbb{N}}\right)^{\mathbb{N}} \approx 2^{\mathbb{N}} \longrightarrow [0,1]$$

Theorem (K.)

- *A uniformly converging subsequence of (f_n) can be computed from a solution of a suitable instance of* $\mathrm{BW}_{\mathrm{weak}}$.
- *Thus there is a uniformly converging subsequence that is low$_2$.*

Theorem (K.)

Over RCA_0 *the following are equivalent*

- $\mathrm{AA}_{\mathrm{weak}}$,
- $\mathrm{BW}_{\mathrm{weak}} + \mathrm{WKL}$.

Program extraction

Theorem (K., Kohlenbach '10)

If $\mathrm{WKL}_0 + \mathrm{BW}_{\mathrm{weak}} + \mathrm{AA}_{\mathrm{weak}} \vdash \forall f\, \exists y\, \mathrm{A}(f, y)$
for quantifier free A*,*
then one can extract from a given proof
a ***primitive recursive*** *(in the sense of Kleene) function(al)* t
such that $\forall f\, \mathrm{A}(f, t(f))$*.*

- "Proof mining"

Upper bound on AA

From a rate of converge of $\left(F(f_{g(n)})\right)_n$ and φ_u one can calculate a rate of convergence for $\left(f_{g(n)}\right)_n$.

Theorem

A uniformly, at the rate 2^{-k}, converging subsequence of (f_n) can be computed from
a solution of a suitable instance of BW *for the space $[0,1]^{\mathbb{N}}$ and φ_u.*

- φ_u can be calculated in $0'$.
- from a solution of a suitable instance of BW one can calculate $0'$.

Theorem

- *For each computable equicontinuous sequence* $f_n\colon [0,1] \to [0,1]$ *there is a* $0'$*-computable* $0/1$*-tree* T*, such that an infinite branch of* T *computes a uniform cluster point, and vice versa.*
- *Over* RCA_0 *the following principles are* ***instance-wise*** *equivalent:*
  - AA,
  - BW,
  - WKL *for* Σ^0_1*-trees.*

Weihrauch lattice

Represent continuous functions $\mathcal{C}([0,1])$ as associates.
Using this we can formulate the Arzelà-Ascoli theorem in terms of the Weihrauch lattice

$$\mathsf{AA} :\subseteq (\mathcal{C}([0,1]))^{\mathbb{N}} \rightrightarrows \mathcal{C}([0,1])$$
$$\mathsf{WAA} :\subseteq (\mathcal{C}([0,1]))^{\mathbb{N}} \rightrightarrows (\mathcal{C}([0,1]))'$$

with $dom(\mathsf{AA}) = dom(\mathsf{WAA}) = \{\, (f_n) \mid (f_n) \text{ equicontinuous} \,\}$ and where $(\mathcal{C}([0,1]))'$ it the derivative of $\mathcal{C}([0,1])$

- $\mathsf{AA} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WKL}'$
- $\mathsf{WAA} \equiv_{\mathrm{W}} \mathsf{WBWT}_{\mathbb{R}}$.
  Notice that we do not need WKL here.

Thank you for your attention!

References

Alexander P. Kreuzer
The cohesive principle and the Bolzano-Weierstraß principle,
Math. Log. Quart. **57** (2011), no. 3, 292–298.

Alexander P. Kreuzer and Ulrich Kohlenbach,
Term extraction and Ramsey's theorem for pairs,
forthcoming J. Symbolic Logic.

Alexander P. Kreuzer,
From Bolzano-Weierstraß to Arzelà-Ascoli,
draft, arXiv:1205.5429.

The cohesive principle

Write $X \subseteq^* Y$ if $X \setminus Y$ is finite.

Definition

- A set X is *cohesive* for a sequence of set $(R_n)_n \subseteq 2^{\mathbb{N}}$ if

$$X \subseteq^* R_n \vee X \subseteq^* \overline{R_n} \qquad \text{for each } n.$$

- The *cohesive principle* (COH) states that for each $(R_n)_n$ there is an infinite cohesive set X.

Theorem (K.)

- *For each sequence $(x_n)_n \subseteq \mathbb{R}$ there exists $(R_n)_n \subseteq 2^{\mathbb{N}}$, such that from an infinite cohesive set for (R_n) one can compute a Cauchy-subsequence of (x_n) and vice versa.*
- $\mathrm{RCA}_0 \vdash \mathrm{BW}_{\mathrm{weak}} \leftrightarrow \mathrm{COH} \wedge B\Sigma^0_2$
  Moreover, this equivalence also holds instance-wise.