Reflection and the fine structure theorem

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Reflection

Let T be a theory.

Reflection is the statement

if ϕ is provable from T, then ϕ is true.

This statement should be understood internally.

Formalization of reflection

Formulas will coded using the standard Gödel numbers.

Definition (Provability predicate)

- Provability predicate:

$$\texttt{Prov}_T(x)$$

- It states that there exists a (code of a) derivation of the formula coded by x in T.
- $\texttt{Prov}_T(x)$ is Σ_1, assuming T has a c.e. axiom set. (We will always assume that.)

Definition (Truth predicate)

- Truth predicate for Π_n-sentences, $\texttt{True}_{\Pi_n}(x)$
- $\texttt{True}_{\Pi_n}(\ulcorner \phi \urcorner) \leftrightarrow \phi$ for $\phi \in \Pi_n$

Formalization of reflection (cont.)

Theorem

$\texttt{True}_{\Pi_n}(x)$ *is* Π_n*-definable.*
(For $n = 0$*,* Δ_1*-definable.)*

Sketch of proof

For $n = 1$ one can take for $\texttt{True}_{\Pi_1}(x)$ the sentence:
If x codes $\forall n \, \phi_0(n)$,
the TM searching for a minimal n with $\neg\phi_0(n)$
does **not terminate**.

Definition (Reflection)

Reflection for a theory T and Π_n statements

$$\texttt{RFN}_T(\Pi_n) :\equiv \texttt{Prov}_T(x) \to \texttt{True}_{\Pi_n}(x).$$

Relation to induction

Let EA := $I\Delta_0 + \text{exp}$.
EA is contained in RCA_0^*.

Theorem (Leivant '83, Ono '87)

$\text{EA} \vdash \texttt{RFN}_{\text{EA}}(\Pi_{n+2}) \leftrightarrow I\Sigma_n \qquad (n \geq 1)$

Sketch of proof

$\rightarrow$: Let $\phi(x) \in \Sigma_n$.
Assume BC : $\phi(0)$ and IS : $\forall x\,(\phi(x) \rightarrow \phi(x+1))$.
Internally, there is a derivation of $\phi(d)$. Apply BC and d-times IS!
$\texttt{RFN}_{\text{EA}}(\Pi_{n+2})$ gives 1

$$\text{BC} \wedge \text{IS} \rightarrow \phi(d)$$

uniformly for all d.

$\leftarrow$: Cut-elimination.

Why reflection?

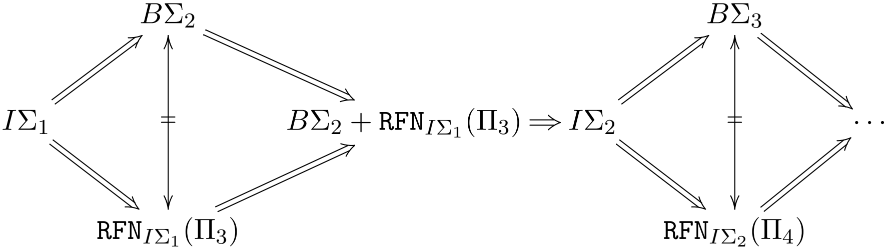
Theorem (partly K., Yokoyama '15)

The following are equivalent over $I\Sigma_1$:

- $\mathtt{RFN}_{I\Sigma_1}(\Pi_3)$,
- *well-foundedness of ω^ω,*
- *Hilbert Basis theorem (Simpson '88),*
- *Formanek/Lawrence Theorem (Hatzikiriakou, Simpson '15)*
- *$P\Sigma_1$ (introduced by Hájek, Paris '86/'87)*
- BME_1 *(introduced by Chong, Slaman, Yang, '14)*
- *The Ackermann function relative to any total function is total.*

- In particular, $\mathtt{RFN}_{I\Sigma_1}(\Pi_3)$ lies strictly between $I\Sigma_1$ and $I\Sigma_2$.
- Observe $\mathtt{RFN}_{I\Sigma_1}(\Pi_3) \equiv \mathtt{RFN}_{\mathtt{RFN}_{EA}(\Pi_3)}(\Pi_3)$. (Iterated reflection!)

Extended Paris-Kirby hierarchy



Reflection and consistency

- Let $\texttt{Con}(T)$ be the consistency of T.
- This can be formulated as $\neg\texttt{Prov}_T(\ulcorner\bot\urcorner)$

Theorem

$\texttt{RFN}_T(\Pi_1)$ *implies* $\texttt{Con}(T)$.

Sketch of Proof

- Suppose $\neg\texttt{Con}(T)$, then $\texttt{Prov}_T(\ulcorner\bot\urcorner)$.
- By $\texttt{RFN}_T(\Pi_1)$ one gets $\texttt{True}_{\Pi_1}(\ulcorner\bot\urcorner)$, i.e., $\bot$. ↯

- Let $\texttt{Con}(\Pi_n + T)$ be the consistency of T plus all Π_n-sentences.
- This can be formulated as $\forall x\, \texttt{True}_{\Pi_n}(x) \rightarrow \neg\texttt{Prov}_T(\tilde{\neg}x)$.

Theorem

$\texttt{RFN}_T(\Pi_{n+1}) \leftrightarrow \texttt{Con}(\Pi_n + T)$.

Existence of models

Theorem (Simpson)

WKL_0^* *proves the completeness theorem, i.e., every consistent theory has a model.*

Model $\mathcal{M}$ is here coded a the set of (Gödel numbers of) sentences true in $\mathcal{M}$.

Existence of models (cont.)

Theorem

Let $n \geq 1$ *and* T *be a theory.*
A model $(\mathcal{M}, \mathcal{S}) \models \mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$ *has an* n*-elementary end extension* $\mathcal{I}$ *satisfying* T*.*

Proof.

- Let $(\mathcal{M}, \mathcal{S})$ be a second-order model with $(\mathcal{M}, \mathcal{S}) \models \mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$.
- Then $(\mathcal{M}, \Delta^0_{n+1}(\mathcal{S})) \models \mathsf{RCA}_0^*$. (Here we use $B\Sigma_{n+1}$.)
- In this model the set of all true Π_n-sentences X exists.
- Extend $(\mathcal{M}, \Delta^0_{n+1}(\mathcal{S}))$ to satisfy WKL_0^*.
- Let $T' := T + X +$ "constants for each element of $\mathcal{M}$".
- By $\mathtt{Con}(\Pi_n + T)$, the theory T' is consistent. By WKL there exists a model of T'. By definition T' is an n-elementary end-extension. □

Existence of models (cont.)

Remark

To make sure that is a true end-extension one can replace T' by $T' + \neg\texttt{Con}(T')$. By Gödel's incompleteness theorem, $T' + \neg\texttt{Con}(T')$ is also consistent.

Remark

In the previous proof we used $B\Sigma_n$ only to get the set of all true Π_n-sentences. If the end-extension $\mathcal{I}$ should satisfy one sentence Π_n-sentence then RCA_0^* is sufficient.

Existence of models (cont.)

Example

Over RCA_0^* the statement $\mathtt{Con}(\Pi_1 + I\Delta_0 + \mathsf{exp})$ proves the totality of superexp, i.e., $n \mapsto \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}}_{n \text{ times}}$.

Proof.

- Let $(\mathcal{M}, \mathcal{S}) \models \mathsf{RCA}_0^* + \mathtt{Con}(\Pi_1 + I\Delta_0 + \mathsf{exp})$.
- Assume that superexp is not total.
  Then there is an $c \in \mathcal{M}$, such that superexp(c) is does not exists.
  In detail, let $\phi(x, y)$ be the Σ_1-formula defining superexp.
  Then $\mathcal{M} \models \forall y\, \neg\phi(c, y)$.
- Note that we have

$$\mathcal{M} \models \exists y\, \phi(0, y), \forall x\ (\exists y\, \phi(c, y) \rightarrow \exists y\, \phi(c + 1, y))\,.$$

...

Proof (continued).

- Let $\mathcal{I}$ be a true end-extension of $\mathcal{M}$ such that $\mathcal{I} \models I\Delta_0 + \mathsf{exp} + \forall y\, \neg\phi(c, y)$.
- We have

$$\mathcal{I} \models \exists y\, \phi(0, y), \forall x\ (\exists y\, \phi(x, y) \rightarrow \exists y\, \phi(x + 1, y))\,.$$

- Working in $\mathcal{M}$, using $I\Delta_0(\mathcal{I})$, we can apply the implication c times and obtain that $\mathcal{I} \models \exists y\, \phi(c, y)$. ↯ □

Iterated reflection

Notation

Let T be a theory.

- $(T)^n_0 := T$,
- $(T)^n_{\alpha+1} := (T)^n_\alpha + \mathtt{RFN}_{(T)^n_\alpha}(\Pi_n)$,
- $(T)^n_\lambda := \bigcup_{\alpha<\lambda} (T)^n_\alpha$.

Example

- $(\mathsf{EA})^3_1 = I\Sigma_1$,
- $(\mathsf{EA})^3_2 = (I\Sigma_1)^3_1 =$ "well-foundedness of ω^ω",
- $(\mathsf{EA})^4_1 = I\Sigma_2$,
- $(\mathsf{EA})^2_1 = I\Delta_0 + \mathsf{exp} + \mathsf{superexp}$.
- $(\mathsf{EA})^2_\omega = \mathsf{PRA}$.

Theorem (Beklemishev '97)

$(EA)^2_\alpha$ is the same as Grzegorczyk arithmetic of level $\alpha + 3$.

Fine structure theorem

Theorem (Schemerl's formula, '79,)

Let $n \geq 1$ and T be a Π_{n+1}-axiomatic extension of EA*.*
$(T)_1^{n+1}$ *is Π_n-conservative over $(T)_\omega^n$. ($n \geq 1$)*

- We prove the case $n = 2$, $T = I\Sigma_1$.
- Proof we proceed by contraposition:
  For $\phi \in \Pi_2$:

  $$\text{If } (T)_\omega^2 \nvdash \phi \quad \text{then} \quad (T)_1^3 \nvdash \phi.$$

- This will be shown by a model construction.
- The construction is a refinement of McAllon '78.

Proof of Schmerl's formula

Given is a non-standard model $\mathcal{I}_0 \models (T)^2_\omega + \neg\phi$.

Goal: Construct a model $\mathcal{M} \models (T)^3_1 + \neg\phi$.

Take a non-standard $b \in \mathcal{I}_0$ such that $\mathcal{I}_0 \models (T)^2_b$.
Let $\mathcal{I}_1$ be a true Π_1-elementary end extension satisfying $(T)^2_{b-1}$, as constructed before.
By construction $\mathcal{I}_0 \models \texttt{Prov}(\ulcorner\psi\urcorner)$ then $\mathcal{I}_1 \models \psi$.
Iterate this construction to get $\mathcal{I}_n \models (T)^2_{b-n}$.
Let $\mathcal{M} := \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$.

Lemma

$\mathcal{M} \models \texttt{RFN}_{I\Sigma_1}(\Pi_3)$

Proof of Schmerl's formula (cont.)

Lemma

$\mathcal{M} \models \mathtt{RFN}_{I\Sigma_1}(\Pi_3)$

Proof.

- Let $\psi = \forall x\, \exists y\, \forall z\, \psi_0(x, y, z)$.
- Suppose $\mathcal{M} \models \mathtt{Prov}(\ulcorner \psi \urcorner)$. Then there is a derivation of ψ in $\mathcal{I}_{k_1}$ for some $k_1 \in \mathbb{N}$.
- Given $c_x \in \mathcal{M}$. Then $c_x \in \mathcal{I}_{k_2}$ for a $k_2 \in \mathbb{N}$.
- $\mathcal{I}_{\max(k_1,k_2)+1} \models \exists y\, \forall z\, \psi_0(c_x, y, z)$.
- In other words, there exists $c_y \in \mathcal{I}_{\max(k_1,k_2)+1}$, s.t. $\mathcal{I}_{\max(k_1,k_2)+1} \models \forall z\, \psi_0(c_x, c_y, z)$.
- By Π_1-elementarity

  $$\mathcal{I}_n, \mathcal{M} \models \psi_0(c_x, c_y, z)$$

  for $n \geq \max(k_1, k_2) + 1$. □

Proof of Schmerl's formula

- Original proof of Schmerl proceeds by comparing well-orders.
- This model-theoretic proof is new.
    - Note that it only works for reasonably strong theories T.
      $I\Sigma_1$ is certainly enough.
      That means $n \geq 2$ or T contains $I\Sigma_1$.
      This is need to extend the model to a model of WKL. Here we use Baire Category theorem for the forcing extension.
      By Simpson '14 the Baire Category theorem is equivalent to $I\Sigma_1$.
    - Natural reflection model.

Question

What is the strength of extending a model $\mathcal{M} \models \mathsf{RCA}_0^*$ to a model of $\mathcal{M} \models \mathsf{WKL}_0^*$?

Construction of a model of SRT^2_2

Theorem (Chong, Slaman, Yang, '14)

$\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ *does not prove* $I\Sigma_2$.

Proof proceeds in two steps:

1. Construct a suitable first-order model.
2. Extend the model to a second-order model using forcing.

Theorem

$\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ *does not prove the well-foundedness of* ω^{ω^2}.

1. Use the model constructed earlier.
2. Extend the model to a second-order model using (a different) forcing.

This theorem follows also from K. Yokoama and L. Patey.

Full fine structure theorem

Theorem (Fine structure theorem, Schmerl '79)

For each $n, k \geq 1$, and all ordinals $\alpha \geq 1$, β, the theory $((\mathsf{EA})^{n+k}_{\alpha})^{n}_{\beta}$ proves the same Π_n-sentences as $(\mathsf{EA})^{n}_{\omega_k(\alpha)\cdot(1+\beta)}$.

Follows from iterations of Schmerl's formula.
For this to work it is sufficient if $n \geq 3$.

Note on the Existence of Models theorem

Theorem

Let $n \geq 1$ and T be a theory.
$\mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$ *proves that there exists an n-elementary ~~end extension~~ $\mathcal{I}$ satisfying T.*

The conclusion of this theorem

There exists a Π_n-elementary model

sometimes also called *reflection*. This theorem say that these two forms of reflection coincide.
For stronger Σ^1_k sets this has been analyzed. This is on the level Π^1_∞-TI. (Friedman, see Simpson's Subsystems of Second Order Arithmetic.)

Summary

- Model-theoretic proof of the fine structure theorem
  - Uses Reverse Mathematics techniques
- Construction for models of well-foundedness of ω^ω.
  - BME_1
  - Hilbert-Basis theorem, Formanek/Lawrence Theorem

Thank you for your attention!