

Program extraction and Ramsey's theorem for pairs

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(joint work with Ulrich Kohlenbach)

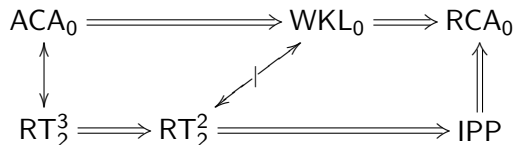
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Ramsey's theorem for pairs and two colors (RT_2^2)

Definition (RT_2^2)

For every coloring $c: [\mathbb{N}]^2 \rightarrow 2$
there exists an infinite set X , such that
 c restricted to $[X]^2$ is constant.



First order consequences of RT_2^2 are known to be between $I\Sigma_2^0$ and $B\Sigma_2^0$.

$$\text{RCA}_0^\omega + \text{RT}_2^2 \vdash \forall x \exists y A(x, y)$$

extract a term t , such that

$$\forall x A(x, t(x)).$$

Here $A(x, y)$ is quantifier-free.

RCA_0^ω is the finite type extension of RCA_0 :

- Sorted into type 0 for \mathbb{N} , type 1 for $\mathbb{N}^{\mathbb{N}}$, type 2 for $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, \dots ,
- contains basic arithmetic $(+, \cdot)$, λ -abstraction,
- quantifier-free axiom of choice for numbers, i.e. the statement that definable functions of natural numbers exist,
- and a recursor R_0 , which provides primitive recursion (for numbers),
- Σ_1^0 -induction.

Program extraction via reduction to arithmetical comprehension

$$\text{RCA}_0^\omega + \text{RT}_2^2 \vdash \forall x \exists y A(x, y)$$

implies

$$\text{RCA}_0^\omega + \Pi_1^0\text{-CA} \vdash \forall x \exists y A(x, y).$$

- Functional interpretation extracts a term t primitive recursive in the bar recursor $B_{0,1}$, such that

$$\forall x A(x, t(x)).$$

- Howard's ordinal analysis of the bar recursor shows that t is provably total relative to Π_∞^0 -induction.

Functional interpretation of RT_2^2

- Formalization of RT_2^2

$$\forall c: [\mathbb{N}]^2 \rightarrow 2 \exists H = \{h_0, h_1, \dots\}$$

$$\forall x, y (x \neq y \rightarrow c(\{h_x, h_y\}) = c(\{h_0, h_1\}))$$

- Functional interpretation:

$$\forall c \quad \exists H^1 \forall x^0, y^0 (x \neq y \rightarrow c(\{h_x, h_y\}) = c(\{h_0, h_1\}))$$

functional interpretation of RT_2^2

$$\forall c \forall X^2, Y^2 \exists H^1 \quad (X(H) \neq Y(H) \rightarrow c(\{h_{X(H)}, h_{Y(H)}\}) = c(\{h_0, h_1\}))$$

- A functional $\mathcal{R}^3(c, X, Y)$ yielding such an H is called a *solution-functional of the functional interpretation of RT_2^2* .

$$\text{RCA}_0^\omega + \text{RT}_2^2 \vdash \forall x \exists y A(x, y)$$

Applying the functional interpretation directly yields a term t primitive recursive in a solution-functional \mathcal{R} of the functional interpretation of RT_2^2 , such that

$$\forall x A(x, t(x))$$

The term t is made of

- $+$, \cdot ,
- the primitive recursor R_0 , i.e.

$$R_0(0, y, f) = y, \quad R_0(x + 1, y, f) = f(R_0(x, y, f), x).$$

- λ -abstraction and
- the solution functional \mathcal{R} .

With coding R_0 is of type 2, \mathcal{R} is of type 3.

\Rightarrow No iteration functional for \mathcal{R} .

\mathcal{R} is only used to build type 2 objects.

Lemma

The term t can be normalized, such that each occurrence of \mathcal{R} is of the form

$$\mathcal{R}(t_0[g], t_1[g], t_2[g])$$

for terms t_i containing only $g: \mathbb{N} \rightarrow \mathbb{N}$ free. This g can be chosen such that it is the same for each application.

How to interpret $\forall g \mathcal{R}(t_0[g], t_1[g], t_2[g])$?

Goal: Show that a restricted use of Π_1^0 -CA suffices to interpret t .

- Full Π_1^0 -CA:

$$\Pi_1^0\text{-CA: } \forall f \exists X \forall k (k \in X \leftrightarrow \forall n f(k, n) \neq 0).$$

- Instance of Π_1^0 -CA:

$$\Pi_1^0\text{-CA}(f): \exists X \forall k (k \in X \leftrightarrow \forall n f(k, n) \neq 0).$$

- $\text{RCA}_0^\omega + \Pi_1^0\text{-CA} \vdash I\Pi_\infty^0$
- $\text{RCA}_0^\omega + [\Pi_1^0\text{-CA}(t) \text{ for all closed terms } t] \vdash \text{light-face-}I\Sigma_2^0$
- For closed terms t :
 $\text{RCA}_0^\omega + \Pi_1^0\text{-CA}(t) \not\vdash I\Sigma_3^0$

Theorem

For all f there exists an f' , such that

$$\begin{aligned} \text{uWKL}_0^\omega \vdash \forall c: [\mathbb{N}]^2 \rightarrow 2 \left(\Pi_1^0\text{-CA}(f'(c)) \right. \\ \left. \rightarrow \exists H (H \text{ infinite and homogeneous for } c \wedge \Pi_1^0\text{-CA}(f(c, H))) \right) \end{aligned}$$

- uWKL_0^ω is roughly RCA_0^ω plus uniform WKL
- Finitely nested uses of instances of RT_2^2 are implied by a suitable single instance of $\Pi_1^0\text{-CA}$ having the same parameters.
- The proof of this theorem is based on Cholak's, Jockusch's and Slaman's proof of the low_2 -ness of RT_2^2 .

Lemma

There exists an f , such that uWKL_0^ω and the functional interpretation of $\forall x \Pi_1^0\text{-CA}(f(g, x))$ proves that

- 1 t is total,
- 2 $\forall x A(x, t(x))$.

Proof.

Normalize t such that it contains only finitely many \mathcal{R} applications of the form $\mathcal{R}(t_0[g], t_1[g], t_2[g])$.

- 1 Replace the applications of \mathcal{R} with the functional interpretation of the last theorem.
- 2 Replace each occurrence of \mathcal{R} in the proof of $\forall x A(x, t(x))$ by this interpretation. □

Theorem (K., Kohlenbach)

$$\text{RCA}_0^\omega + \text{RT}_2^2 \vdash \forall x \exists y A(x, y)$$

then one can extract a term t provably total in $\text{RCA}_0^\omega + I\Sigma_2^0$ such that

$$\text{RCA}_0^\omega + I\Sigma_2^0 \vdash \forall x A(x, t(x))$$

Proof.

- Use functional interpretation to extract a term t primitive recursive in \mathcal{R} , such that $\forall x A(x, t(x))$.
- Normalize t . Replace the occurrence of \mathcal{R} to obtain a proof in $\text{uWKL}_0^\omega +$ the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$.
- Solve the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$ with a *single use* of the bar recursor $B_{0,1}$.
- Kohlenbach's elimination of WKL and Howard's analysis of the bar recursor yield the result. □

Theorem (K., Kohlenbach)

$$\text{RCA}_0^\omega + I\Sigma_2^0 + \text{RT}_2^2 \vdash \forall x \exists y A(x, y)$$

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$$\text{RCA}_0^\omega + I\Sigma_2^0 \vdash \forall x A(x, t(x))$$

Proof.

- Use functional interpretation to extract a term t primitive recursive in \mathcal{R} and the recursor R_1 , such that $\forall x A(x, t(x))$.
- Normalize t . Replace the occurrence of \mathcal{R} and R_1 to obtain a proof in $\text{uWKL}_0^\omega +$ the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$.
- Solve the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$ with a *single use* of the bar recursor $B_{0,1}$.
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Theorem (K., Kohlenbach)

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$$\text{RCA}_0^\omega + I\Sigma_2^0 \vdash \forall x A(x, t(x))$$

Proof.

- Use functional interpretation to extract a term t primitive recursive in \mathcal{R} and the recursor R_1 and the restricted bar recursor Φ_{WKL} for WKL , such that $\forall x A(x, t(x))$.
- Normalize t . Replace the occurrence of \mathcal{R} and R_1 to obtain a proof in $\text{uWKL}_0^\omega +$ the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$. Use Howard's analysis of the restricted bar recursor to interpret Φ_{WKL} .
- Solve the functional interpretation of $\Pi_1^0\text{-CA}(f(g, x))$ with a *single use* of the bar recursor $B_{0,1}$.
- Kohlenbach's elimination of WKL and Howard's analysis of the bar recursor yield the result. □

Proofwise low

This is a general concept:

Call a principle P of the form

$$\forall X \exists Y P'(X, Y)$$

proofwise low over a system \mathcal{T} if

for all f exists an f' , such that

$$\mathcal{T} \vdash \forall X \left(\Pi_1^0\text{-CA}(f'(X)) \rightarrow \exists Y \left(P'(X, Y) \wedge \Pi_1^0\text{-CA}(f(X, Y)) \right) \right).$$

Theorem (K., Kohlenbach)

If P is proofwise low over WKL_0^ω and P' is Π_1^0 then

$$\text{WKL}_0^\omega + I\Sigma_2^0 + P$$

is Π_3^0 -conservative and admits term extraction over the system

$$\text{RCA}_0^\omega + I\Sigma_2^0.$$

Let $WKL_0^{\omega*}$ be the system WKL_0^{ω} where $I\Sigma_1^0$ and R_0 is replaced by $I\Sigma_0^0$, 2^x and bounded primitive recursion.

Theorem (K.)

If P is proofwise low over WKL_0^{ω} and P' is Π_1^0 then*

$$WKL_0^{\omega} + B\Sigma_2^0 + P$$

is Π_3^0 -conservative and admits extraction of primitive recursive terms over the system

$$RCA_0^{\omega}.$$

Proof.

Use a refinement of Howard's ordinal analysis to terms in elementary arithmetic ($G_{\infty}A^{\omega}$). □

The chain antichain principle

Definition

Let the *chain antichain principle* (CAC) be that statement the each partial order over \mathbb{N} possesses an infinite chain or an infinite antichain.

Lemma (Cholak, Jockusch, Slaman)

$$\text{RCA}_0 \vdash \text{RT}_2^2 \rightarrow \text{CAC}$$

The chain antichain principle

Theorem (Chong, Slaman, Yang)

$$\text{RCA}_0 + B\Sigma_2^0 + \text{CAC}$$

is Π_1^1 -conservative over

$$\text{RCA}_0 + B\Sigma_2^0.$$

Theorem (K.)

The chain antichain principle is proofwise low over $\text{WKL}_0^{\omega*}$.

Hence

$$\text{WKL}_0^{\omega} + B\Sigma_2^0 + \text{CAC}$$

is Π_3^0 -conservative over

$$\text{RCA}_0^{\omega}.$$

Moreover **primitive recursive** terms can be extracted for $\forall\exists$ sentences.

Connections to the Bolzano-Weierstraß principle

The chain antichain principle implies the following variant of the Bolzano-Weierstraß principle:

Each bounded sequence in \mathbb{R} contains a Cauchy-subsequence.




Compare to the reverse mathematics formulation of the Bolzano-Weierstraß principle:

Each bounded sequence in \mathbb{R} contains a converging subsequence, i.e. a Cauchy-subsequence with Cauchy-rate 2^{-n} .

Theorem (K.)

- RCA_0 proves that the strong cohesive principle ($\text{COH} + B\Sigma_2^0$) is equivalent to our variant of the Bolzano-Weierstraß principle.
- ADS is equivalent to the stronger statement that each sequence of real numbers contains a monotone subsequence.

- We introduced the notion of *proofwise low*.
- Program extraction and conservativity results for proofwise low principles.
- Application to RT_2^2 and CAC:
 - Extraction of terms of Ackermann type resp. primitive recursive terms.
 - New proof for the facts that RT_2^2 does not imply more than Ackermannian growth and that CAC does not imply Σ_2^0 -induction.
 - Program extraction for a variant of the Bolzano-Weierstraß principle

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