On the Uniform Computational Content of the Baire Category Theorem

Alexander P. Kreuzer

joint work with

Vasco Brattka and Matthew Hendtlass

National University of Singapore

Dagstuhl Seminar 15392, September 2015

Baire category theorem

Let X be a complete metric space.

Theorem (Baire category theorem)

Let $(A_n)_{n\in\mathbb{N}}$ be closed nowhere dense subsets of X.

$$\bigcup_{n\in\mathbb{N}} A_n \subsetneq X$$

Representation of closed sets

- **negative information**, $\mathcal{A}_-(X)$, ϕ_-,
  A is the complement of an open set given as an enumeration of basic open balls
- **positive information**, $\mathcal{A}_+(X)$, ϕ_+,
  A is the closure of an enumeration of points in X.
- **set of cluster points**, $\mathcal{A}_*(X)$, ϕ_*,
  A is the set of cluster points of an enumeration of points in X.

Example: Closed Choice

Let X metric space.

Definition

$$\mathsf{C}_X :\subseteq \mathcal{A}_{\square}(X) \rightrightarrows X$$
$$A \mapsto A$$

- **positive information** ($\square = +$),
  C_X is trivial.
- **set of cluster points** ($\square = *$),
  C_X is the same as finding a cluster point (similar as Bolzano-Weierstraß theorem)
- **negative information** ($\square = -$),
  right formulation, non-continuous.

Baire category theorem

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Formulate as computational problem:

- **BCT_0** Given $(A_n)_{n\in\mathbb{N}}$ closed nowhere dense. There is an $x \in X \setminus \bigcup_{n\in\mathbb{N}} A_n$.

  $$\mathsf{BCT}_0 :\subseteq \mathcal{A}_{-}(X)^{\mathbb{N}} \rightrightarrows X$$

- **BCT_1** Given $(A_n)_{n\in\mathbb{N}}$ closed, such that $\bigcup_{n\in\mathbb{N}} A_n = X$. There is an index n such that A_n is somewhere dense.

  $$\mathsf{BCT}_1 :\subseteq \mathcal{A}_{-}(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}$$

BCT_2, BCT_3 are defined like BCT_0 and BCT_1 but with positive input.

Overview over the strength of BCT

		classical reverse mathematics
BCT_0	computable	RCA_0
BCT_1	equivalent to $\mathsf{C}_{\mathbb{N}}$ implies Banach inverse mapping theorem, etc.	RCA_0
BCT_2	computability theoretic version related to 1-generic, forcing, BCT*II*	$\Pi^0_1\mathsf{G}$
BCT_3	equivalent to cluster point problem	$\Pi^0_1\mathsf{G}$

Space X has to be perfect (no isolated points.) E.g., $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$.

In classical reverse mathematics (Brown, Simpson '93)
$\mathsf{RCA}_0 + \mathsf{BCT}\mathit{II} \vdash$ "Banach inverse mapping theorem".

BCT on non-perfect space X

If X has isolated points.

Proposition

$$\mathsf{BCT}_2 \equiv_{\mathsf{sW}} id_0 \qquad \mathsf{BCT}_3 \equiv_{\mathsf{sW}} id_{\mathbb{N}}$$

In particular, BCT_2, BCT_3 are computable in this case.

BCT on perfect spaces

Theorem (Brattka, Hendtlass, K.)

BCT_i *for a perfect polish space* X *is* sW-*equivalent to* BCT_i *for* $\mathbb{N}^{\mathbb{N}}$.

Proof.

- BCT_i for X is reducible to BCT_i for $\mathbb{N}^{\mathbb{N}}$:
  - Let $\delta\colon \mathbb{N}^{\mathbb{N}} \to X$ be the Cauchy representation.
  - Take $\delta\mathsf{BCT}_i(\delta^{-1}(A_n))$.
- BCT_i for $2^{\mathbb{N}}$ is reducible to BCT_i for X:
  - By perfectness there is an embedding $\iota : 2^{\mathbb{N}} \to X$ with computable inverse. (Brattka, Gheradi '08)
  - $A \subseteq 2^{\mathbb{N}}$ nowhere dense $\Rightarrow \iota(A)$ nowhere dense.
  - Take $\iota^{-1}(\mathsf{BCT}_i((\iota(A_n))_n))$.
- BCT_i for $\mathbb{N}^{\mathbb{N}}$ is reducible to BCT_i for $2^{\mathbb{N}}$:
  - Embed $\mathbb{N}^{\mathbb{N}}$ into $2^{\mathbb{N}}$ via $p \mapsto 1^{p(0)}01^{p(1)}01^{p(2)}0\cdots$.
  - Range is (c.e.) comeager.

□

Consider now only $X = \mathbb{N}^{\mathbb{N}}$.

Theorem (Brattka '01, Brattka, Gherardi '11)

- $\mathsf{C}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}_1$,
- $\mathsf{CL}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}_3 \equiv_{\mathsf{sW}} \mathsf{BCT}'_1$.

Theorem (Brattka, Hendtlass, K.)

- BCT_0, BCT_2 *are densely realized.*
  In particular $\mathsf{C}_2 \not\leq_{\mathsf{W}} \mathsf{BCT}_0, \mathsf{BCT}_2$.
- $\mathsf{BCT}'_0 \equiv_{\mathsf{sW}} \mathsf{BCT}_2$.

Proof of $\mathsf{BCT}_2 \equiv_{\mathsf{sW}} \mathsf{BCT}_0'$ and $\mathsf{BCT}_3 \equiv_{\mathsf{sW}} \mathsf{BCT}_1'$

Proposition

$id_{+-} \colon \mathcal{A}_+(X) \to \mathcal{A}_-(X) \leq_{\mathsf{sW}} \lim$

Gives $\mathsf{BCT}_2 = \mathsf{BCT}_0 \circ \mathrm{id}_{+-} \leq_{\mathsf{sW}} \mathsf{BCT}_0'$ and $\mathsf{BCT}_3 \leq_{\mathsf{sW}} \mathsf{BCT}_1'$.

Proposition (Brattka, Gherardi, Marcone '12)

$id \colon \mathcal{A}_*(X) \to \mathcal{A}_-(X)'$ is a computable isomorphism.

Proposition

There is an $M :\subseteq \mathcal{A}_*(X) \rightrightarrows \mathcal{A}_+(X)$ such that,

- $M(A) \subseteq \{B \colon A \subseteq B\}$
- *A nowhere dense $\Rightarrow$ $B \in M(A)$ nowhere dense. (X perfect)*

The mapping

$$\mathcal{A}_-(X)' \xrightarrow{id} \mathcal{A}_*(X) \xrightarrow{M} \mathcal{A}_+(X)$$

gives $\mathsf{BCT}_0' \leq_{\mathsf{sW}} \mathsf{BCT}_2$ and $\mathsf{BCT}_1' \leq_{\mathsf{sW}} \mathsf{BCT}_3$.

1-generic

A point $p \in 2^{\mathbb{N}}$ is 1-generic relative to q if it meets or avoids any c.e. open set U_i^q, i.e.,

$$\exists w \sqsubseteq p \ \left(w2^{\mathbb{N}} \subseteq U_i^q \quad \text{or} \quad w2^{\mathbb{N}} \cap U_i^q = \emptyset \right).$$

Equivalently: $p \notin \partial U_i^q$

Theorem

$\mathsf{BCT}_0 \leq_{\mathsf{sW}} 1\text{-}\mathsf{GEN} \leq_{\mathsf{sW}} \mathsf{BCT}_2$

Proof.

1. For nowhere dense A, $A = \partial A = \partial A^c$.

$$\mathsf{BCT}_0((A_i)_i) = 2^{\mathbb{N}} \setminus \bigcup_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} (2^{\mathbb{N}} \setminus \partial A_i^c)$$

   Now $A_i^c = U_j^q$ for a suitable j. Thus, $\mathsf{BCT}_0 \leq_{\mathsf{sW}} 1\text{-}\mathsf{GEN}$.

2. Use $\mathsf{BCT}_2 \equiv_{\mathsf{sW}} \mathsf{BCT}_0'$ and compute ∂U_i^q in the limit. □

1-generic (cont.)

Theorem

$\mathsf{BCT}_0 <_{\mathrm{sW}} 1\text{-}\mathsf{GEN} <_{\mathrm{sW}} \mathsf{BCT}_2$

(The implications are strict.)

Proof sketch.

1. Sufficient to use a weakly 1-generic in the previous proof.
   Apply the fact that there are weakly 1-generics that are not 1-generic.
   $\mathsf{BCT}_0 \leq_{\mathrm{sW}} 1\text{-}\mathsf{weakGEN} <_{\mathrm{sW}} 1\text{-}\mathsf{GEN}$.
2. (Uniform) Theorem of Kurtz shows that $1\text{-}\mathsf{GEN} \leq_{\mathrm{sW}} \mathsf{WWKL}'$.[a]

   Lemma of Kučera shows that WWKL' can be realized such that its output is low for Ω.

   There is a computable p such that $\mathsf{BCT}_2(p)$ is not low for Ω.

   Thus, $\mathsf{BCT}_2 \not\leq_{\mathrm{W}} \mathsf{WWKL}'$. □

[a] Actually, $(1-*)$-WWKL'

Π^0_1G

Definition (Π^0_1G, classical reverse math)

Let $D_i \subseteq 2^{<\mathbb{N}}$ be a sequence of dense, uniformly Π^0_1-set.
There is a set $G \subseteq 2^{\mathbb{N}}$ meeting each D_i, i.e., $\exists s \in D_i\,(s \sqsubseteq G)$.

Π^0_1G related to forcing constructions.
Formulation in the Weihrauch lattice: Model properties of D_i using a suitable representation.

Definition

$\phi_{\#}(p) = A :\Longleftrightarrow \phi_{-}(p) = D$ and $A = 2^{\mathbb{N}} \setminus \bigcup_{w \in D} w2^{\mathbb{N}}$,
where $D \subseteq 2^{<\mathbb{N}}$.

Definition (Π^0_1G, Weihrauch version)

$$\Pi^0_1\mathsf{G} :\subseteq \mathcal{A}_{\#}(2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}} \qquad (A_n)_n \mapsto \bigcap 2^{\mathbb{N}} \setminus A_n,$$

with $\mathrm{dom}(\Pi^0_1\mathsf{G}) := \{(A_i)_i \mid A_i^{\circ} = \emptyset\}$.

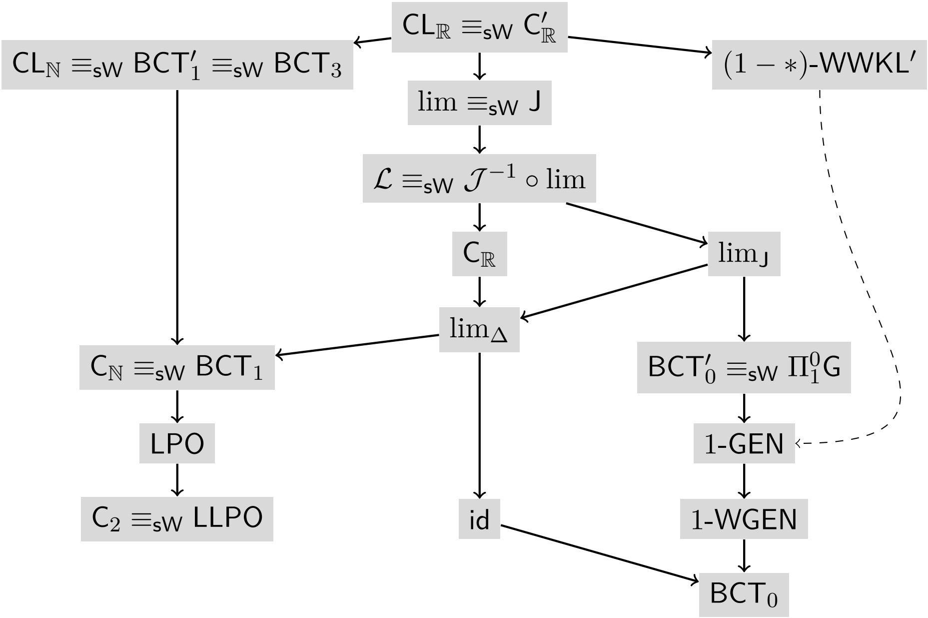
Π^0_1G (cont.)

Proposition

$\mathrm{id}\colon \mathcal{A}_-(2^{\mathbb{N}})' \to \mathcal{A}_\#(2^{\mathbb{N}})$ *is a computable isomorphism.*

Corollary

$\Pi^0_1\mathsf{G} \equiv_{\mathsf{sW}} \mathsf{BCT}_0' \equiv_{\mathsf{sW}} \mathsf{BCT}_2$


$\mathsf{CL}_{\mathbb{R}} \equiv_{\mathsf{sW}} \mathsf{C}'_{\mathbb{R}}$
$\mathsf{CL}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}'_1 \equiv_{\mathsf{sW}} \mathsf{BCT}_3$
$(1-*)\text{-}\mathsf{WWKL}'$
$\lim \equiv_{\mathsf{sW}} \mathsf{J}$
$\mathcal{L} \equiv_{\mathsf{sW}} \mathcal{J}^{-1} \circ \lim$
$\mathsf{C}_{\mathbb{R}}$
$\lim_{\mathsf{J}}$
$\lim_{\Delta}$
$\mathsf{C}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}_1$
$\mathsf{BCT}'_0 \equiv_{\mathsf{sW}} \Pi^0_1\mathsf{G}$
LPO
$1\text{-}\mathsf{GEN}$
$\mathsf{C}_2 \equiv_{\mathsf{sW}} \mathsf{LLPO}$
id
$1\text{-}\mathsf{WGEN}$
BCT_0

Summary

- Different ways of writing Baire category theorem as forall-exists statement are natural.
- Basically 2 different variants.
- Calculus characterization of $\Pi^0_1 G$.

Thank you for your attention!

Bibliography

V. Brattka, M. Hendtlass, A. Kreuzer
On the Uniform Computational Content of Baire Category Theorem

V. Brattka, M. Hendtlass, A. Kreuzer,
On the Uniform Computational Content of Computability Theory
arXiv:1501.00433