The Bolzano-Weierstraß principle and the cohesive principle

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The Bolzano-Weierstraß principle

A real number is a sequence of rational numbers with Cauchy-rate 2^{-n}.

Definition

(BW):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ has a cluster point.

Equivalently, every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence (y_n) with Cauchy-rate 2^{-n}, i.e. with

$$\forall n \, \forall i, j \geq n \, \left(|y_i - y_j| < 2^{-n} \right).$$

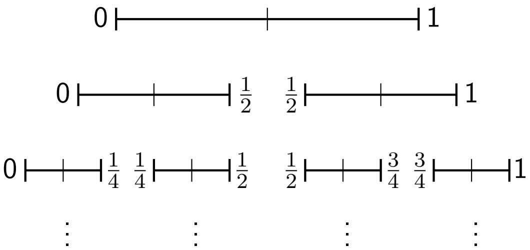
Definition

($\mathsf{BW}_{\mathsf{weak}}$):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence $(y_n)_n$, i.e.

$$\forall n \, \exists k \, \forall i, j \geq k \, \left(|y_i - y_j| < 2^{-n} \right).$$

Computing BW

Assume that $(x_n)_n \subseteq [0,1] \cap \mathbb{Q}$.
Goal: Construct a subsequence (y_n) with $\forall n \, \forall i, j \geq n \; (|y_i - y_j| < 2^{-n})$.



- bi-partition argument
- $y_n :=$ { next element of (x_n) in the n-th partition chosen

- The partitions form a Π^0_2-0/1-tree.
- This is a Π^0_1-0/1-tree in $0'$.
- WKL relativized to $0'$ yields an infinite branch and therefore computes the sequence of partitions.

Theorem (Kohlenbach '98, Kohlenbach, Safarik '10, K. '10)

- *For each computable sequence* (x_n) *there is a* $0'$*-computable* $0/1$*-tree* T*, such that an infinite branch of* T *computes a cluster point, and vice versa.*
- *Over* RCA_0 *the principles* BW *and* WKL *for* Σ^0_1*-trees are* ***instance-wise*** *equivalent.*

By the low basis theorem:

Corollary

BW *has for computable instances a solution* low *relative to* $0'$*, i.e. the first Turing jump of a solution is computable in* $0''$*.*

Computing BW_{weak}

Assume that $(x_n)_n \subseteq [0,1] \cap \mathbb{Q}$.
Goal: Construct a subsequence (y_n) with $\forall n\, \exists k\, \forall i,j \geq k\ (|y_i - y_j| < 2^{-n})$ **and** compute the Turing jump of (y_n).

It is clear that

$$\Phi_e^{(y_n)_n}\downarrow \quad \text{iff} \quad \exists k\, \Phi_e^{(y_n)_{n<k}}\downarrow.$$

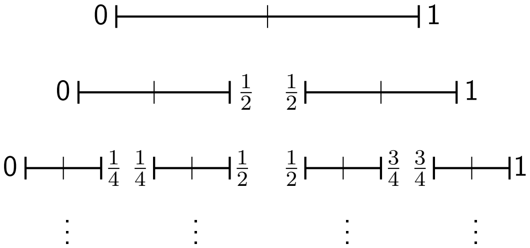
Suppose that $(y_n)_{n<m}$ is an initial segment that has already been computed. Deciding, whether there is an extension $(y_n)_{n<l}$, such that

$$\Phi_e^{(y_n)_{n<l}}\downarrow$$

can be done in $0'$.

Computing BW_{weak}

Assume that $(x_n)_n \subseteq [0,1] \cap \mathbb{Q}$.
Goal: Construct a subsequence (y_n) with $\forall n \, \exists k \, \forall i,j \geq k \; (|y_i - y_j| < 2^{-n})$
and compute the Turing jump of (y_n).



- bi-partition argument
- now add at each step not only one element but finitely many elements of the chosen interval to (y_n).

- Let $(y_n)_{n<m}$ be the initial segment of (y_n) computed up to the k-th step.
  At the k-th step extend this to $(y_n)_{n<l}$ by elements in the k-th chosen interval, such that $\Phi_k^{(y_n)_{n<l}}\!\downarrow$, if possible.
- Then extend this by another element of the interval.

Theorem (K.)

For each bounded, computable sequence (x_n)
there is a Cauchy-subsequence (y_n),
such that (y_n) *and* $(y_n)'$ *are computable in*
a Turing degree that contains infinite branches of $0'$*-computable 0/1-trees.*

Corollary

$\mathsf{BW}_{\mathsf{weak}}$ *has* low_2 *solutions,*
i.e. $(y_n)''$ *is computable in* $0''$.

Proof.

$$(y_n)' \leq_T 0' + \mathsf{WKL} \quad \Longrightarrow \quad (y_n)'' \leq_T 0'' \qquad \square$$

$\mathsf{BW}_{\mathsf{weak}}$ does not compute $0'$
and is therefore strictly weaker than BW.

The cohesive principle

Write $X \subseteq^* Y$ if $X \setminus Y$ is finite.

Definition

- A set X is *cohesive* for a sequence of set $(R_n)_n \subseteq 2^{\mathbb{N}}$ if

$$X \subseteq^* R_n \vee X \subseteq^* \overline{R_n} \qquad \text{for each } n.$$

- The *cohesive principle* (COH) states that for each $(R_n)_n$ there is an infinite cohesive set X.

Theorem (K.)

- *For each sequence $(x_n)_n \subseteq \mathbb{R}$ there exists $(R_n)_n \subseteq 2^{\mathbb{N}}$, such that from an infinite cohesive set for (R_n) one can compute a Cauchy-subsequence of (x_n) and vice versa.*
- $\mathsf{RCA}_0 \vdash \mathsf{BW}_{\mathsf{weak}} \leftrightarrow \mathsf{COH} \wedge B\Sigma^0_2$
  Moreover, this equivalence also holds instance-wise.

Theorem

1. COH *and hence also* $\mathsf{BW}_{\mathsf{weak}}$ *do not compute solutions to* WKL *in general. (Cholak, Jockusch, Slaman '01)*
2. *There are instance of these principles which have no* low *solutions. (Jockusch, Stephan '93)*

2. refines a result by Le Roux and Ziegler.

Proof of the low_2-ness of $\mathsf{BW}_{\mathsf{weak}}$ is a streamlined version of the low_2-ness of COH (Jockusch, Stephan '93).

Theorem (Chong, Slaman, Yang '10)

$\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ *and thus* $\mathsf{RCA}_0 + \mathsf{BW}_{\mathsf{weak}}$
are Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2$.

Theorem (K., Kohlenbach '10)

If $\mathsf{WKL}_0 + \mathsf{BW}_{\mathsf{weak}} \vdash \forall f\, \exists y\, \phi(f, y)$
for quantifier free ϕ,
then one can extract from a given proof
a ***primitive recursive*** *(in the sense of Kleene) function(al)* t
such that $\forall f\, \phi(f, t(f))$.

- "Proof mining"

Bolzano-Weierstraß in the weak topology

We consider the Hilbert space $\ell_2 = (\mathbb{R}^{\mathbb{N}}, \langle\cdot,\cdot\rangle)$.
An element of ℓ_2 is given by a Cauchy-sequence $(w_n)_n$ of finite dimensional and rational approximations, i.e. $w_n \in \mathbb{Q}^{<\mathbb{N}}$, with Cauchy-rate 2^{-n} with respect to $\|\cdot\|$.

Definition

(weak-BW): Every $\|\cdot\|$-bounded sequence $(x_n) \subseteq \ell_2$ has a weak cluster point x, i.e. for a subsequence $(x_{f(n)})$ we have
$\forall y \in \ell_2 \lim_{n\to\infty}\langle y, x_{f(n)}\rangle = \langle y, x\rangle$.

Bolzano-Weierstraß in the weak topology (cont.)

Theorem (K.)

1. *For each bounded sequence $(x_n) \subseteq \ell_2$ there is a weak cluster point x computable in $0''$.*
2. *There is a bounded and computable sequence $(x_n) \subseteq \ell_2$, such that each weak cluster point of it computes $0''$.*
3. *Over* RCA$_0$ *the principles* Π^0_2-CA *and* weak-BW *are* ***instance-wise*** *equivalent.*

Proof of 1.

- Use BW to compute a cluster point $y = (y_1, y_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ of (x_n) in the topology of $\mathbb{R}^{\mathbb{N}}$.
  Computable in $0' +$ WKL (BW for $\mathbb{R}$ is the same as for $\mathbb{R}^{\mathbb{N}}$.)
- $\sum_{i<k} y_i e_i$ converges monotonically in ℓ_2 to a weak cluster point of (x_n).
  Computable in the jump. □

Bolzano-Weierstraß in the weak topology (cont.)

Using the classification of the Bolzano-Weierstraß principle for metric spaces in the Weihrauch lattice (Brattka, Gherardi, Marcone '11) we can also obtain the following.

Theorem

$$\text{weak-BW} \equiv_{\mathrm{sW}} \lim^{(2)}$$

The *weak presentation* of an element $x \in \ell_2$ is a sequence of finite dimensional approximations $(w_n)_n$ and a norm-bound b such that

$$\lim_{n\to\infty} w_n = x \quad \text{and} \quad \forall n \, \|w_n\|_2 < b$$

Question (Brattka)

What is the computation strength of the Bolzano-Weierstraß principle for elements of ℓ_2 given in the ***weak presentation****?*

Question

What is the computational strength of the Bolzano-Weierstraß principle for elements of $\mathbb{R}$ given by ***slowly converging sequences*** *as input and output?*

For $\mathrm{BW}_{\mathrm{weak}}$ the input is given by sequences with Cauchy-rate 2^{-n} and the output by slowly converging sequence.

Summary

- BW is equivalent to WKL for $0'$-computable trees.
- $\mathrm{BW}_{\mathrm{weak}}$ is equivalent to COH.
  - Hence, it does not imply $0'$.
  - It admits extraction of primitive recursive terms.
- weak-BW is equivalent to $0''$.

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