On the Uniform Computational Content of Computability Theory

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joint work with

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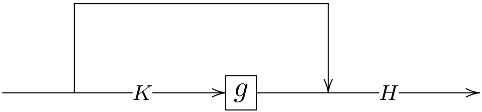
Weihrauch lattice

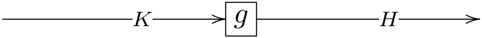
Weihrauch reduction: Let $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$f \leq_{\mathsf{W}} g$ iff $\exists K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ computable $(f = H\langle \mathrm{id}, gK \rangle)$



Strong variant:

$f \leq_{\mathsf{sW}} g$ iff $\exists K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ computable $(f = HgK)$



Mutlivalued functions

Let $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be multivalued.

Definition

$F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ realizes f iff

$$F(x) \in f(x) \qquad \text{for all } x \in \mathrm{dom}(f).$$

Write $F \vdash f$.

$f \leq_{\mathrm{W}} g$ if

$$\exists K, H\, \forall G \vdash g\ \Big(H\langle \mathrm{id}, GK\rangle \vdash f\Big).$$

Same for $\leq_{\mathrm{sW}}$.

Represented Spaces

Spaces X, Y are represented by surjective function $\delta_X, \delta_Y :\subseteq \mathbb{N}^{\mathbb{N}} \to X, Y$. A realizer $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ to a multivalued function on represented spaces $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ is function such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Examples

Closed Choice:
$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, X \mapsto X$

$\mathsf{C}_2 \equiv_{\mathsf{sW}} \mathsf{LLPO}$

Compositional products:
$f * g := \max\{f_0 \circ g_0 \mid f_0 \leq_{\mathsf{W}} f, g_0 \leq_{\mathsf{W}} g\}$

Algebraic operations:
Product $f \times g$, parallelization $\hat{f}$, etc.

Weak König's lemma

WKL *Weak König's lemma*
Every infinite 0/1-tree, has an infinite branch.

DNC_k *Diagonal non-computable function*
For every $p \in 2^{\mathbb{N}}$, there exists a diagonal non-computable function $f\colon \mathbb{N} \to k$, i.e., $f(n) \neq \phi_n^p(n)$.

PA *Completion of Peano arithmetic*
For every p there is a Turing-degree d containing a completion of each p-computable theory.

Theorem (classical)

Computationally (non-uniform) the following are equivalent:

- WKL,
- DNC_k *for any* $k \in \mathbb{N}$,
- PA.

$\text{DNC}_{\mathbb{N}}$ is weaker.

WKL in the Weihrauch lattice

Theorem

$$\mathsf{WKL} \equiv_{\mathsf{sW}} \widehat{\mathsf{LLPO}}$$

Definition (ACC_X, all or co-unique choice)

$$\mathsf{ACC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

and $\mathrm{dom}(\mathsf{ACC}_X) := \{A \in \mathcal{A}_-(X) \colon |X \setminus A| \leq 1 \text{ and } A \neq \emptyset\}$.

Theorem (Weihrauch, '92)

$$\mathsf{ACC}_{\mathbb{N}} <_{\mathsf{W}} \mathsf{ACC}_{n+1} <_{\mathsf{W}} \mathsf{ACC}_n <_{\mathsf{W}} \mathsf{ACC}_2 \equiv_{\mathsf{sW}} \mathsf{LLPO}$$

Theorem (Brattka, Hendtlass, K.)

$$\mathsf{DNC}_X \equiv_{\mathsf{sW}} \widehat{\mathsf{ACC}_X}$$

In particular, $\mathsf{WKL} \equiv_{\mathsf{sW}} \widehat{\mathsf{LLPO}} \equiv_{\mathsf{sW}} \mathsf{DNC}_2$.

WKL in the Weihrauch lattice (cont.)

Theorem (Brattka, Hendtlass, K.)

$$\mathsf{ACC}_n \not\leq_{\mathsf{W}} \mathsf{DNC}_{n+1}$$

$$\mathsf{DNC}_{\mathbb{N}} <_{\mathsf{W}} \mathsf{DNC}_{n+1} <_{\mathsf{W}} \mathsf{DNC}_n <_{\mathsf{W}} \mathsf{DNC}_2 \equiv_{\mathsf{sW}} \mathsf{WKL}$$

Turing degrees as represented spaces

Let $[p] := \{q \in \mathbb{N}^{\mathbb{N}} \mid p \equiv_{\mathrm{T}} q\}$

Definition (Turing degrees, representation)

- $\mathcal{D} := \{\ [p] \mid p \in \mathbb{N}^{\mathbb{N}}\ \}$,
- $\delta_{\mathcal{D}} \colon \mathbb{N}^{\mathbb{N}} \to \mathcal{D}, p \mapsto [p]$.

Observation

Turing degrees are invariant under finite modification of its members.

$\delta_{\mathcal{D}}^{-1}(d)$ for $d \in \mathcal{D}$, is <u>dense</u>.

We call such spaces <u>densely realized</u>.

Densely realized

A multi-valued map $f :\subseteq X \rightrightarrows Y$ is called densely realized, if

$\{\ F(p) \mid F \vdash f\ \}$ is dense for all $p \in \mathrm{dom}(f\delta_X)$.

Proposition

If Y as above is densely realized, f is densely realized.

Theorem

If f is densely realized, then

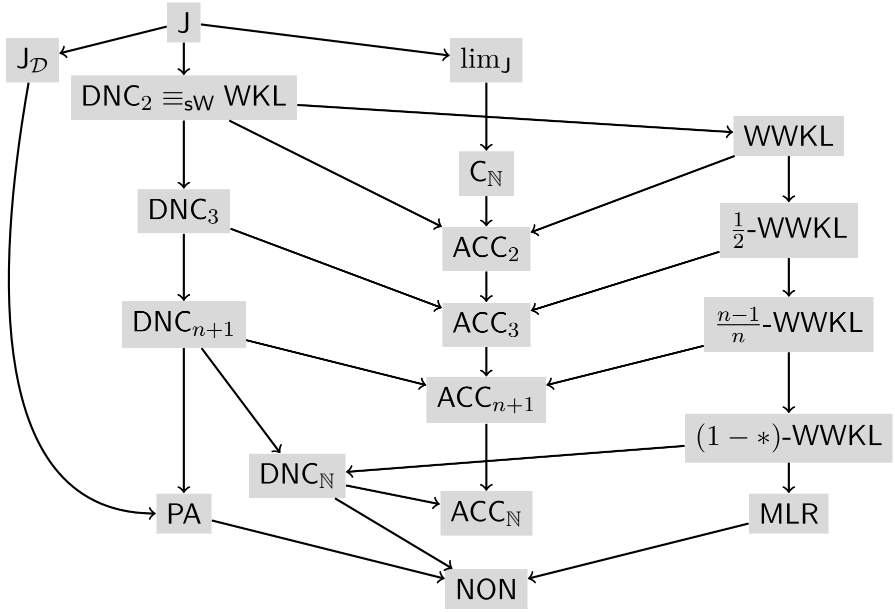
$$\mathsf{ACC}_{\mathbb{N}} \not\leq_{\mathrm{W}} f. \tag{1}$$

Proof: Continuity! $\square$.

Property (1) is called ω-indiscriminative.

Corollary

PA $:\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$ *is ω-indiscriminative. Thus,* $\mathsf{DNC}_{\mathbb{N}} \not\leq_{\mathrm{W}}$ PA.


J
$J_{\mathcal{D}}$
$\lim_J$
$DNC_2 \equiv_{sW}$ WKL
WWKL
$C_{\mathbb{N}}$
DNC_3
ACC_2
$\frac{1}{2}$-WWKL
DNC_{n+1}
ACC_3
$\frac{n-1}{n}$-WWKL
ACC_{n+1}
$(1-*)$-WWKL
$DNC_{\mathbb{N}}$
PA
$ACC_{\mathbb{N}}$
MLR
NON

Other principles considered

- Weak weak König's lemma and Martin-Löf randomness
- Jump inversion theorem

  $\mathsf{JIT} : d \mapsto \{a \mid a' = d \cup \emptyset'\}$,

  $\mathsf{JIT} <_{\mathsf{sW}} c_{\emptyset'} \times id$
- Kleene-Post theorem

Relates to (refines) other approaches:

Theorem (Relation to Medevdev reducibility)

For $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$,

$$f \leq_{\mathsf{W}} g \Longrightarrow$$

$$\forall p \in \mathrm{dom}(f) \cap \mathsf{COMP}\ \exists q \in \mathrm{dom}(g) \cap \mathsf{COMP}\ \Big(f(p) \leq_{\mathsf{M}} g(q)\Big).$$

Our analysis of DNC_k refines work by Cenzer, Hinmann in Medevdev lattice.

Indiscriminative

Definition

$f :\subseteq X \rightrightarrows Y$ is called

- indiscriminative if $\mathsf{LLPO} \not\leq_{\mathrm{W}} \mathsf{f}$,
- ω-indiscriminative if $\mathsf{ACC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{f}$.

Are indiscriminative principles useful?

No: Obviously do not compute much.
Probably, the reason why most of recursion theory does not show up in analysis.
WKL is an exception.

Yes: I will present some examples.

Reasons for being indiscriminative

- Computational weakness,
- Continuity,
- Densely realized,
  - Range is densely realized as space
    - Turing degrees $\mathcal{D}$,
    - Derived spaces
  - Definition of the principle

Examples

- Weak Bolzano Weierstraß principle

$$\mathsf{WBWT}_{\mathbb{R}} :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}'$$

- Cohesive principle, (variants of) Baire category theorem

Cohesive principle

Definition

- Let $(R_i)_{i\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$. A set $X \in 2^{\mathbb{N}}$ is called cohesive if
  - X is infinite.
  - $X \subseteq^* R_i$ or $X \subseteq^* \overline{R_i}$ for all i.
- COH $:\subseteq (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$

Proposition

COH *is densely realized.*

Proof: By definition. □

Corollary

- COH *is ω-indiscriminative.*
- WKL $\not\leq_W$ COH,
- $\mathrm{DNC}_{\mathbb{N}}$, MLR $\not\leq_W$ COH.

Cohesive principle and weak Bolzano-Weierstraß

Theorem (K. '11)

$\mathsf{WBWT}_{\mathbb{R}} \equiv_{\mathsf{W}} \mathsf{COH}$.

Note: non-strong Weihrauch equivalence. There is a variant of $\mathsf{SBWT}_{\mathbb{R}}$ for which strong equivalence holds.

Proposition

$\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathsf{sW}} \lim * \mathsf{WBWT}_{\mathbb{R}} \equiv_{\mathsf{sW}} \lim * \mathsf{COH}$

Theorem (Brattka, Gherardi, Marcone '12; K.)

$\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathsf{sW}} \mathsf{WKL}' \equiv_{\mathsf{sW}} \mathsf{WKL} * \lim$

Cohesive principle and weak Bolzano-Weierstraß

$$\lim * \mathsf{COH} \equiv_{\mathrm{W}} \mathsf{WKL}' \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$$

Is COH optimal? Is there a weaker principle such that

$$\lim * \boxed{?} \equiv_{\mathrm{W}} \mathsf{WKL}'$$

Yes, COH is optimal.

Theorem

$\mathsf{COH} \equiv_{\mathrm{W}} \lim \rightarrow \mathsf{WKL}'$

Side info on $\rightarrow$, (Brattka, Pauly '14)

$f \rightarrow g := \min\{h \mid g \leq_{\mathrm{W}} f * h\}$.

- $f \rightarrow g$ is the weakest oracle for f needed to compute g.
- Exists always.

Algebraic characterization of COH

Cohesive degrees

Degree variant of COH:

$$[\mathsf{COH}] :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$$

Jump for degrees:

$$\mathsf{J}_{\mathcal{D}} \colon d \mapsto \{d'\}$$

Theorem (Jockusch, Stephan '93 (essentially))

$[\mathsf{COH}] \equiv_{\mathsf{W}} \mathsf{J}_{\mathcal{D}}^{-1} \circ \mathsf{PA} \circ \mathsf{J}_{\mathcal{D}}$.

Note:

$$\mathsf{COH} \not\leq_{\mathsf{W}} \lim^{-1} * \mathsf{WKL} * \lim$$

Theorem

$[\mathsf{COH}] \equiv_{\mathsf{W}} (\lim \to \mathsf{PA}') \equiv_{\mathsf{W}} (\mathsf{J}_{\mathcal{D}} \to \mathsf{PA}')$

Baire category theorem

Let X be a complete metric space.

Theorem (Baire category theorem)

Let $(A_i)_{i\in\mathbb{N}}$ be closed nowhere dense subsets of X.

$$\bigcup_{i\in\mathbb{N}} A_i \subsetneq X$$

Formulate as computational problem:

- BCT_0 Given $(A_i)_{i\in\mathbb{N}}$ closed nowhere dense. There is an $x \in X \setminus \bigcup_{i\in\mathbb{N}} A_i$.

  $$\mathsf{BCT}_0 :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows X$$

- BCT_1 Given $(A_i)_{i\in\mathbb{N}}$ closed, such that $\bigcup_{i\in\mathbb{N}} A_i = X$. There is an index i such that A_i is somewhere dense.

  $$\mathsf{BCT}_1 :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}$$

BCT_2, BCT_3 are defined like BCT_0 and BCT_1 but with positive input.

Baire Category theorem (cont.)

		classical reverse mathematics
BCT_0	computable	RCA_0
BCT_1	computable with finitely many mind changes $\mathsf{C}_{\mathbb{N}}$	$\mathsf{RCA}_0 + \mathsf{BCTII}$
BCT_2	computability theoretic version related to 1-generic, forcing	$\Pi^0_1\mathsf{G}$
BCT_3	equivalent to cluster point problem	ACA_0

Space X has to be perfect (no isolated points.) E.g., $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$.

Non perfect space:

Proposition

$$\mathsf{BCT}_2 \equiv_{\mathsf{sW}} id_0 \qquad \mathsf{BCT}_3 \equiv_{\mathsf{sW}} id_{\mathbb{N}}$$

In particular BCT_2, BCT_3 are computable in this case.

Baire Category theorem

Theorem (Brattka, Hendtlass, K.)

BCT_i *for a perfect polish space* X *is strong-Weihrauch equivalent to* BCT_i *for* $\mathbb{N}^\mathbb{N}$*.*

Consider now only $X = \mathbb{N}^\mathbb{N}$.

Theorem (Brattka '01, Brattka, Gherardi '11)

- $\mathsf{C}_\mathbb{N} \equiv_{\mathsf{sW}} \mathsf{BCT}_1$,
- $\mathsf{CL}_\mathbb{N} \equiv_{\mathsf{sW}} \mathsf{BCT}_3 \equiv_{\mathsf{sW}} \mathsf{BCT}_1'$.

BCT_1, BCT_3 are discriminative.

Theorem (Brattka, Hendtlass, K.)

- BCT_0, BCT_2 *are densely realized and hence* ω*-indiscriminative.*
- $\mathsf{BCT}_0' \equiv_{\mathsf{sW}} \mathsf{BCT}_2$.

Proof of $\mathsf{BCT}_2 \equiv_{\mathsf{sW}} \mathsf{BCT}'_0$ and $\mathsf{BCT}_3 \equiv_{\mathsf{sW}} \mathsf{BCT}'_1$

Representations:

negative information	$\mathcal{A}_-, \phi_-$	Enumerate balls in complement
positive information	$\mathcal{A}_+, \phi_+$	Closure of points
cluster point	$\mathcal{A}_*, \phi_*$	Cluster points of points

Proposition

$id_{+-}\colon \mathcal{A}_+(X) \to \mathcal{A}_-(X) \leq_{\mathsf{sW}} \lim$

Gives $\mathsf{BCT}_2 \leq_{\mathsf{sW}} \mathsf{BCT}'_0$ and $\mathsf{BCT}_3 \leq_{\mathsf{sW}} \mathsf{BCT}'_1$.

Proposition (Brattka, Gherardi, Marcone '12)

$id\colon \mathcal{A}_*(X) \to \mathcal{A}_-(X)'$ *is a computable isomorphism.*

Proposition

There is an $M :\subseteq \mathcal{A}_*(X) \rightrightarrows \mathcal{A}_+(X)$ *such that,*

- $M(A) \subseteq \{B\colon A \subseteq B\}$
- A *nowhere dense* $\Rightarrow B \in M(A)$ *nowhere dense. (X perfect)*

1-generic

A point $p \in 2^{\mathbb{N}}$ is 1-generic relative to q if it meets or avoids any c.e. open set U_i^q, i.e.,

$$\exists w \sqsubseteq p \quad \left(w2^{\mathbb{N}} \subseteq U_i^q \quad \text{or} \quad w2^{\mathbb{N}} \cap U_i^q = \emptyset \right).$$

Equivalently: $p \notin \partial U_i^q$

Theorem

$\mathsf{BCT}_0 \leq_{\mathrm{sW}} 1\text{-}\mathsf{GEN} \leq_{\mathrm{sW}} \mathsf{BCT}_2$

Proof.

1. For nowhere dense A, $A = \partial A = \partial A^c$.

$$\mathsf{BCT}_0 = 2^{\mathbb{N}} \setminus \bigcup_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} (2^{\mathbb{N}} \setminus \partial A_i^c)$$

   Now $A_i^c = U_j^q$ for a suitable j. Thus, $\mathsf{BCT}_0 \leq_{\mathrm{sW}} 1\text{-}\mathsf{GEN}$.

2. Use $\mathsf{BCT}_2 \equiv_{\mathrm{sW}} \mathsf{BCT}_0'$ and compute $(U_i^q)^c$ in the limit. □

1-generic (cont.)

Theorem

$\mathsf{BCT}_0 <_{\mathsf{sW}} 1\text{-}\mathsf{GEN} <_{\mathsf{sW}} \mathsf{BCT}_2$
(The implications are strict.)

Proof sketch.

1. Sufficient to use a weakly 1-generic in the previous proof.
   Apply the fact that there are weakly 1-generics that are not 1-generic.
2. (Uniform) Theorem of Kurtz shows that $1\text{-}\mathsf{GEN} \leq_{\mathsf{sW}} \mathsf{WWKL}'$.[a]

   Lemma of Kučera shows that WWKL' can be realized such that its output is low for Ω.

   There is a computable p such that $\mathsf{BCT}_2(p)$ is not low for Ω.

   Thus, $\mathsf{BCT}_2 \not\leq_{\mathsf{W}} \mathsf{WWKL}'$.

□

[a] Actually, $(1-*)$-WWKL'

Π^0_1G

Definition (Π^0_1G, classical reverse math)

Let $D_i \subseteq 2^{<\mathbb{N}}$ be a sequence of dense, uniformly Π^0_1-set. There is a set $G \subseteq 2^{\mathbb{N}}$ meeting each D_i, i.e., $\exists s \in D_i \, (s \sqsubseteq G)$.

Π^0_1G related to forcing constructions.
Formulation in the Weihrauch lattice: Model properties of D_i using a suitable representation

Definition

$\phi_{\#}(p) = D :\Longleftrightarrow \phi_{-}(p) = E$ and $A = 2^{\mathbb{N}} \setminus \bigcup_{w \in E} w2^{\mathbb{N}}$,
where $E \subseteq 2^{<\mathbb{N}}$.

Definition (Π^0_1G, Weihrauch version)

$$\Pi^0_1\mathsf{G} :\subseteq \mathcal{A}_{\#}(2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}} \qquad (D_i)_i \mapsto \bigcap 2^{\mathbb{N}} \setminus D_i,$$

with $\mathrm{dom}(\Pi^0_1\mathsf{G}) := \{(A_i)_i \mid A_i^{\circ} = \emptyset\}$.

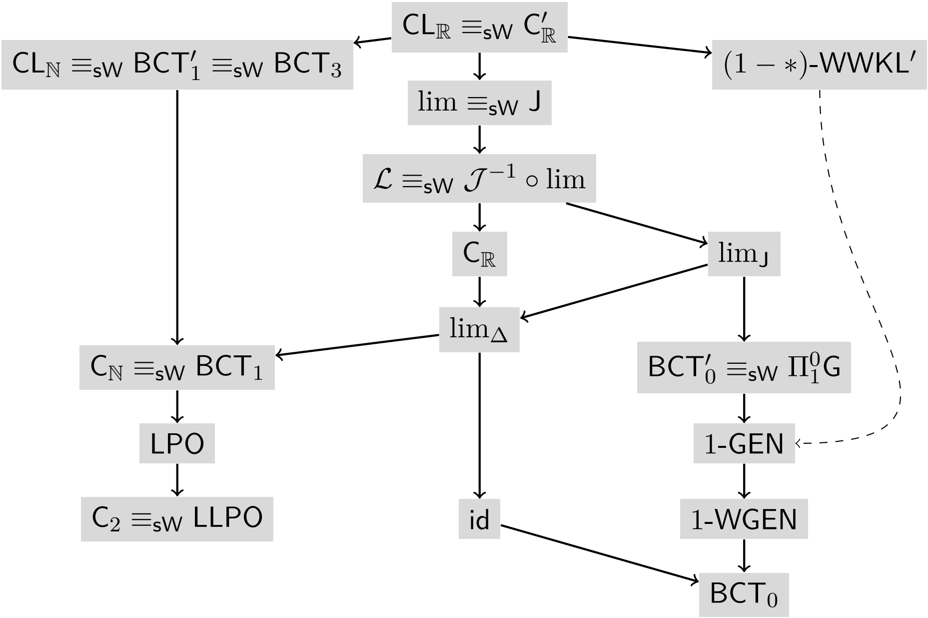
Π^0_1G (cont.)

Proposition

$\mathrm{id}\colon \mathcal{A}_-(2^{\mathbb{N}})' \to \mathcal{A}_\#(2^{\mathbb{N}})$ *is a computable isomorphism.*

Corollary

$\Pi^0_1\mathsf{G} \equiv_{\mathsf{sW}} \mathsf{BCT}'_0 \equiv_{\mathsf{sW}} \mathsf{BCT}_2$



$\mathsf{CL}_{\mathbb{R}} \equiv_{\mathsf{sW}} \mathsf{C}'_{\mathbb{R}}$
$\mathsf{CL}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}'_1 \equiv_{\mathsf{sW}} \mathsf{BCT}_3$
$(1-*)$-WWKL′
$\lim \equiv_{\mathsf{sW}} \mathsf{J}$
$\mathcal{L} \equiv_{\mathsf{sW}} \mathcal{J}^{-1} \circ \lim$
$\mathsf{C}_{\mathbb{R}}$
$\lim_{\mathsf{J}}$
$\lim_{\Delta}$
$\mathsf{C}_{\mathbb{N}} \equiv_{\mathsf{sW}} \mathsf{BCT}_1$
$\mathsf{BCT}'_0 \equiv_{\mathsf{sW}} \Pi^0_1\mathsf{G}$
LPO
1-GEN
$\mathsf{C}_2 \equiv_{\mathsf{sW}} \mathsf{LLPO}$
id
1-WGEN
BCT_0

What more do we see in the Weihrauch lattice?

- Characterization of DNC_k as parallelization of weak omnisience principle ACC_k.
- Algebraic characterization of $\mathsf{COH} \equiv_{\mathsf{W}} \lim \rightarrow \mathsf{WKL}'$.
- Calculus characterization of $\Pi^0_1\mathsf{G}$.

Thank you for your attention!

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