Ramsey's theorem for pairs and provable recursive functions

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Outline

1. Introduction
   - Reverse Mathematics
   - Strength of Ramsey's theorem
2. Theorem
   - Elimination of Skolem functions for monotone formulas
   - Comparison of proof-techniques
3. Recent work

Ramsey's Theorem

Let $[\mathbb{N}]^k$ be the set of unordered k-tuples of natural numbers.
A n-*coloring* of $[\mathbb{N}]^k$ is a map of $[\mathbb{N}]^k$ into $\mathbf{n}$.

Definition (RT^k_n)

For every n-coloring of $[\mathbb{N}]^k$
exists an infinite *homogeneous* set $H \subseteq \mathbb{N}$
(i.e. the coloring is constant on $[H]^k$).

Lemma

$\mathrm{RT}^k_n \leftrightarrow \mathrm{RT}^k_{n'}$, *for all* $n, n' \in \mathbb{N} \setminus \{1\}$.

$\mathrm{RT}^k_{<\infty}$ is defined as $\forall n\, \mathrm{RT}^2_n$.

Reverse Mathematics

Seeks to find which axioms over seconder order arithmetic are needed to prove theorems. Usually consider the *big five* subsystems of second order arithmetic:

RCA_0	Σ^0_1-IA + recursive comprehension
	1st order part: Σ^0_1-induction
	2nd order part: recursive sets
WKL_0	RCA_0 + Weak Königs Lemma
	1st order part: like RCA_0
	2nd order part: low sets
ACA_0	RCA_0 + arithmetic comprehension
	1st order part: arithmetic induction
	2nd order part: arithmetic sets (includes WKL)
ATR_0	ACA_0 + arithmetical transfinite recursion
Π^1_1-CA_0	ACA_0 + Π^1_1-comprehension

Weaker systems:

RCA^*_0	QF-IA + recursive comprehension + exponential function
WKL^*_0	RCA^*_0 + Weak Königs Lemma

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Strength of Ramsey's theorem

- RT^1_n is derivable in pure logic.
- $\mathrm{RT}^1_{<\infty}$ is the *infinite pigeonhole principle*.

  $$\mathrm{RT}^1_{<\infty} \leftrightarrow \Pi^0_1\text{-CP}$$

  Π^0_3-conservative over RCA_0. (Hirst, Friedman)
  Especially $\mathrm{RT}^1_{<\infty}$ cause only primitive recursive growth.
- $$\mathrm{RT}^k_n \leftrightarrow \mathrm{ACA}_0$$

  for $k \geq 3$ and $n \geq 2$. (Simpson)

Strength of Ramsey's theorem for pairs

Theorem (Hirst)

- $\mathrm{RT}^2_2 \rightarrow \Pi^0_1\text{-CP}$
- $\mathrm{RT}^2_{<\infty} \rightarrow \Pi^0_2\text{-CP}$

Theorem (Jockusch)

There exists a computable coloring, which has no in $0'$ computable infinite homogeneous set. Especially $\mathrm{WKL}_0 \nvdash \mathrm{RT}^2_2$.

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Especially $\mathrm{WKL}_0 \nvdash \mathrm{RT}^2_2$.

Theorem (Cholak, Jockusch, Slaman)

- *Every computable coloring has an infinite homogeneous H set which is low_2, i.e. $H'' \leq_T 0''$.*
- $\mathrm{RCA}_0 + \Sigma^0_2\text{-IA} + \mathrm{RT}^2_2$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + \Sigma^0_2\text{-IA}$.
- $\mathrm{WKL}_0 + \Sigma^0_3\text{-IA} + \mathrm{RT}^2_{<\infty}$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + \Sigma^0_3\text{-IA}$.

Question (Cholak, Jockusch, Slaman)

- *Does* RT^2_2 *imply* $\Sigma^0_2\text{-IA}$ *or the totality of the Ackermann-Function?*
- *Does* $\mathrm{RT}^2_{<\infty}$ *imply* $\Sigma^0_3\text{-IA}$*?*

We show

$$\mathrm{WKL}^*_0 + \text{instances of } \mathrm{RT}^2_2 + \text{instances of } \Sigma^0_1\text{-IA}$$

does not prove the totality of the Ackermann-Function.

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Main Result

For a schema $\mathcal{S}$ let $\mathcal{S}^-$ denote the schema restricted to instances which only have number parameters.

Theorem (K., Kohlenbach)

For every fixed n

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^- + {\mathrm{RT}^2_n}^-$$

is

- Π^0_2*-conservative over* PRA,
- Π^0_3*-conservative over* $\mathrm{PRA} + \Sigma^0_1\text{-}\mathrm{IA}$ *and*
- Π^0_4*-conservative over* $\mathrm{PRA} + \Pi^0_1\text{-}\mathrm{CP}$.

Grzegorczyk Arithmetic in all finite types ($G_\infty A^\omega$)

Corresponds to the (full) Grzegorczyk hierarchy.
Contains

- quantifier free induction,
- bounded primitive recursion with function parameters,
- all primitive recursive functions,
- but not all primitive recursive functionals.
  The function iterator is not contained.

Remark

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Remark

The system RCA_0^* *can be embedded into* $\mathrm{G}_3\mathrm{A}^\omega + \mathrm{QF}\text{-}\mathrm{AC}$.

$G_\infty A^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$

Lemma

$G_\infty A^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$ *proves*

- $\Pi^0_1\text{-IA}^-$, $\Sigma^0_1\text{-IA}^-$,
- $\Pi^0_1\text{-AC}^-$, $\Pi^0_1\text{-CP}^-$,
- $\Sigma^0_1\text{-WKL}^-$,
- BW^- *(instances of Bolzano-Weierstrass).*

All these principles cannot be nested.

Proof

Theorem

For every fixed n

$$G_\infty A^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^- + {\text{RT}^2_n}^-$$

is

- Π^0_2*-conservative over* PRA,
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Proof.

1. Show $G_\infty A^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$ proves ${\text{RT}^2_n}^-$.
2. Use elimination of Skolem functions to obtain conservation result. □

Reduction step

Analyze Erdős' and Rado's proof of RT^2_n based on full König's Lemma.

Theorem (K., Kohlenbach)

$$\mathrm{G}_\infty\mathrm{A}^\omega + \Pi^0_1\text{-}\mathrm{IA}^- \vdash \Sigma^0_1\text{-}\mathrm{WKL}^- \to {\mathrm{RT}^2_n}^-$$

for every fixed n.

Corollary

$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF}\text{-}\mathrm{AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^-$ *proves* ${\mathrm{RT}^2_n}^-$, *for every fixed n.*

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Elimination of Skolem functions for monotone formulas

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Elimination of Skolem functions for monotone formulas

Let $\mathcal{T}^\omega := \mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL}$.

Theorem (Kohlenbach)

For every closed term ξ:

$$\mathcal{T}^\omega \vdash \forall f : \mathbb{N}^\mathbb{N}\ \big(\Pi^0_1\text{-}\mathrm{CA}(\xi(f)) \rightarrow \exists x \in \mathbb{N}\, A_{qf}(f,x)\big)$$

$\Rightarrow$ *one can extract a (Kleene-)primitive recursive functional Φ s.t.*

$$\mathrm{PRA}^\omega \vdash \forall f : \mathbb{N}^\mathbb{N}\, A_{qf}(f, \Phi(f)).$$

Experience from proof-mining shows that many theorems from mathematics can be proved in this system.

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Theorem (K., Kohlenbach)

Let ξ_1 and ξ_2 be closed terms and n be fixed.

$$\mathcal{T}^\omega \vdash \forall f \left(\Pi^0_1\text{-}\mathrm{CA}(\xi_1(f)) \wedge \forall k\, \mathrm{RT}^2_n(\xi_2(f, k)) \rightarrow \exists x \in \mathbb{N}\, A_{qf}(f, x)\right)$$

$\Rightarrow$ *one can extract a (Kleene-)primitive recursive functional Φ s.t.*

$$\mathrm{PRA}^\omega \vdash \forall f : \mathbb{N}^\mathbb{N}\, A_{qf}(f, \Phi(f))$$

Bound on n

Theorem (Jockusch)

The exists a primitive recursive sequence of instances of $\mathrm{RT}^2_{<\infty}$ proving the totality of the Ackermann-function.

Theorem (K., Kohlenbach)

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^- \nvdash \mathrm{RT}^{2-}_{<\infty}$$

Results

Theorem (K., Kohlenbach)

For every fixed n a primitive recursive sequence of instance of RT^2_n does not prove the totality of the Ackermann-function. Especially

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^- + {\mathrm{RT}^2_n}^- \nvdash \Sigma^0_2\text{-}\mathrm{IA}.$$

Remark

This yields in the language of RCA_0:

$$\mathrm{WKL}^*_0 + \Pi^0_1\text{-}\mathrm{CA}^- + {\mathrm{RT}^2_n}^- \nvdash \Sigma^0_2\text{-}\mathrm{IA}$$

Comparison of proof-techniques

To prove that $\mathcal{T}$ is arithmetical- / Π^0_2-conservative over $\mathcal{T}'$

reverse mathematics

- Take an arbitrary model $\mathcal{M}$ of $\mathcal{T}'$.
- Extend $\mathcal{M}$ to a model of $\mathcal{T}$ without changing the arithmetical sentences (e.g. with syntactic forcing).
- The conservation results follows then from the completeness theorem.

proof interpretation

- Take an arbitrary proof of a Π^0_2-statement

$$\mathcal{T} \vdash \forall x \, \exists y \, A(x,y).$$

- Obtain using the functional interpretation a term t realizing y A, i.e. $\forall x \, A(x, tx)$.
- Eliminate the Skolem functions resulting from axioms in $\mathcal{T} \setminus \mathcal{T}'$.
- $\mathcal{T}' \vdash \forall x \, A(x, \tilde{t}x)$

Recent results

Can the elimination of Skolem functions for monotone formulas be applied to nested instances of RT^2_2 or to full RT^2_2?

Split RT^2_2 into COH and SRT^2_2 (see CJS).

Theorem (details need to be check)

For all closed terms ξ_1, ξ_2 there exists a closed term ϕ such that

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} \vdash \forall f$$
$$\left(\Pi^0_1\text{-}\mathrm{CA}(\phi f) \rightarrow \exists g \left(g \textit{ is a solution to } \mathrm{COH}(\xi_1 f) \wedge \Pi^0_1\text{-}\mathrm{CA}(\xi_2 f g)\right)\right)$$

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Combining these theorem, proves nested instances of RT^2_2. A realizing term for a statement proven in this system is provably total in $\mathrm{PRA}^\omega + \Sigma^0_2\text{-IA}$.

Methods used: monotone functional interpretation, bar recursion (Spector), uniform weak Königs Lemma, normalization

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References

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