PRIMITIVE RECURSION AND THE CHAIN ANTICHAIN PRINCIPLE

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Abstract. Let the chain antichain principle (CAC) be the statement that each partial order on $\mathbb{N}$ possesses an infinite chain or an infinite antichain. Chong, Slaman and Yang recently proved using forcing over non-standard models of arithmetic that CAC is Π^1_1-conservative over $\mathrm{RCA}_0 + \Pi^0_1$-CP and so in particular that CAC does not imply Σ^0_2-induction. We provide here a different purely syntactical and constructive proof of the statement that CAC (even together with WKL) does not imply Σ^0_2-induction. In detail we show that $\mathrm{WKL}^\omega_0 + \mathrm{CAC}$ is Π^0_2-conservative over PRA and that one can extract primitive recursive realizers for such statements. Moreover, our proof is finitary in the sense of Hilbert's program.

CAC implies that every sequence of real numbers has a monotone subsequence. This Bolzano-Weierstraß like principle is commonly used in proofs. Our result makes it possible to extract primitive recursive terms from such proofs.

Our proof is based on the techniques we develop together with Kohlenbach in [21]. In the course of the proof we refine Howard's ordinal analysis of bar recursion.

We also discuss the Erdős-Moser principle, which —taken together with CAC— is equivalent to RT^2_2.

Let the chain antichain principle (CAC) be the statement that every partial order on $\mathbb{N}$ contains either an infinite chain or an infinite antichain. This principle is a consequence of Ramsey's theorem for pairs (RT^2_2). The principle RT^2_2 states that for each coloring of unordered pairs of $\mathbb{N}$ there exists an infinite subset of $\mathbb{N}$ on which this coloring is constant. The chain antichain principle has been studied in the reverse mathematics of partial orders. Lately it has received much attention in the context of the classification of RT^2_2 and in particular in the context of determining the strength of the first order consequences of RT^2_2. It is known that RT^2_2 implies Π^0_1-CP and that its first order consequences are implied by Σ^0_2-IA but it is not known where between these principles the first order consequences of RT^2_2 lie, see [4, 11]. Chong, Slaman, Yang in [5] recently proved that CAC is Π^1_1-conservative over $\mathrm{RCA}_0 + \Pi^0_1$-CP which implies that CAC does not yield Σ^0_2-induction. This result is remarkable since forcing over ω-models —which is usually used to obtain such conservativity results— is not applicable to obtain conservativity over Π^0_1-CP, see [11, §6]. Chong, Slaman, Yang use instead a forcing over non-standard models

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of arithmetic. This result raises the question whether one can extend it to obtain the conservativity of RT^2_2 or a least gain insights in the properties of principles that do imply Π^0_1-CP but not Σ^0_2-IA like CAC.

We provide here a different, purely syntactical and constructive proof of the fact that CAC does not imply Σ^0_2-induction. We show that CAC even together with WKL is Π^0_2-conservative over PRA. Furthermore, we provide a method for the extraction of primitive recursive realizing functionals for sentences of the form $\forall f\, \exists y\, A_{qf}(f,y)$ that are provable using CAC + WKL. (This means that we extract a primitive recursive functional φ with $\forall f\, A_{qf}(f, \varphi f)$.) Our proof is based on the techniques from [21], where we developed a method to extract terms of Ackermann type from proofs using RT^2_2 and primitive recursive terms from proofs using the cohesive principle and the atomic model theorem.

In [21] we introduced the notion *proofwise low*. Roughly speaking, this notion covers the computational content of low_2-ness but also keeps track of the induction used in the proof. A Π^1_2-principle P of the form

$$\forall X\, \exists Y\, P'(X,Y) \tag{1}$$

is proofwise low over a system, say WKL^ω_0, if for each term φ a term ξ exists such that

$$\mathrm{WKL}^\omega_0 \vdash \forall X\, \left(\Pi^0_1\text{-CA}(\xi X) \rightarrow \exists Y\, \left(P'(X,Y) \wedge \Pi^0_1\text{-CA}(\varphi XY)\right)\right). \tag{2}$$

Here $\Pi^0_1\text{-CA}(t) :\equiv \exists f\, \forall n\, (f(n) = 0 \leftrightarrow \forall x\, t(n,x) = 0)$ and the ω superscript at WKL_0 indicates that we use the finite type variant of WKL_0. This means that WKL^ω_0 is not sorted into two types for $\mathbb{N}$ and subsets of $\mathbb{N}$, but into countable many types for $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ etc. This system is conservative over WKL_0, see [18].

If one takes for φ in (2) the characteristic term of universal Turing predicate $\Phi^{X,Y}_n(n)\uparrow$ and notes that one can take for ξ also the Turing predicate $\Phi^X_n(n)\uparrow$, one has that in a degree $d \gg X'$ — this takes account of WKL — one can compute Y and Y'. From this follows that P has low_2 solutions. In [21] we showed that for principles P of the form (1) where P' is Π^0_3 and which are proofwise low over WKL^ω_0 the system $\mathrm{WKL}^\omega_0 + \Sigma^0_2\text{-IA} + \mathrm{P}$ is Π^0_3-conservative over $\mathrm{RCA}^\omega_0 + \Sigma^0_2\text{-IA}$ and that one can extract realizing terms from Π^0_2 sentences. We, moreover, showed that RT^2_2 is proofwise low over a refinement of WKL^ω_0 for which this result still holds. This provides a different purely proof-theoretic proof of the well known results from Cholak, Jockusch, Slaman in [4].

Model-theoretically speaking the rough idea behind this proof is the following. Take a first order model $\mathcal{N} = \langle N, +, \cdot, 0, 1\rangle$ that satisfies Σ_2-induction. We would like to show that one could extend $\mathcal{N}$ to an L_2-model of RT^2_2 and Σ^0_2-induction. For this consider the extension of $\mathcal{N}$ to an L_2-model $\mathcal{M} = \langle N, \mathbb{X}, +, \cdot, 0, 1\rangle$ by all Δ_2-definable sets of $\mathcal{N}$. This model satisfies Δ^0_1-CA and, since $\Sigma_1(\Delta_2)$-induction is equivalent to Σ_2-induction without parameters, also Σ^0_1-IA. Thus $\mathcal{M} \models \mathrm{RCA}_0$. The model $\mathcal{M}$ can be extended to an model of WKL_0, see [24, Theorem IX.2.1]. We will also call this model $\mathcal{M}$.

Now consider the extension of $\mathcal{N}$ to another L_2-model $\mathcal{M}' = \langle N, \mathbb{X}', +, \cdot, 0, 1\rangle$ where $\mathbb{X}' = \{X \subseteq N \mid X' \in \mathbb{X}\}$. Clearly $\mathcal{M}' \subseteq \mathcal{M}$. By the lowness property (2) for $X = \emptyset$ and $\varphi = \Phi^{X,Y}_n(n)\uparrow$ interpreted in $\mathcal{M}$ the set $\mathbb{X}'$ is closed under applications of P. Hence $\mathcal{M}' \models \mathrm{P}$, which is in our case $\mathcal{M}' \models \mathrm{RT}^2_2$. The model $\mathcal{M}'$ also satisfies Σ^0_2-induction and Δ^0_1-CA for formulas containing *not more than one* set parameter.

Unfortunately one cannot show that for two sets $X, Y \in \mathbb{X}'$ that $X \oplus Y \in \mathbb{X}'$. Therefore $\mathcal{M}' \nvDash \mathrm{RCA}_0$.

In [21] we did a detailed bookkeeping of the uses of comprehension and the parameters that are involved along a proof of a $\forall\exists$-statement in a system like $\mathrm{WKL}_0^\omega + \mathrm{RT}_2^2$. In order to have access to this information we first applied a functional interpretation. With this we could circumvent the problem occurring in the sketch.

Let RCA_0^* be RCA_0 where Σ_1^0-IA is replaced by QF-IA and the exponential function (see [24, X.4]) and let $\mathrm{RCA}_0^{\omega *}$ be the corresponding finite type variant. In [21] we also showed that for principles P which are proofwise low over $\mathrm{WKL}_0^{\omega *}$ (under an additional uniformity assumption) the system $\mathrm{WKL}_0^\omega + \Pi_1^0\text{-CP} + \mathrm{P}$ is Π_3^0-conservative over RCA_0^ω. (In [21] this is called proofwise low in sequence.) This is sufficient for the cohesive principle (COH). However for most principles this uniformity assumption do not hold. In particular, RT_2^2 and CAC do not satisfy it, see Proposition 2 and Remark 3 in [21].

In this paper we close this gap and show that for each principle P which is proofwise low over $\mathrm{WKL}_0^{\omega *}$ the system $\mathrm{WKL}_0^\omega + \Pi_1^0\text{-CP} + \mathrm{P}$ is Π_3^0-conservative over RCA_0^ω and that one can extract primitive recursive realizing terms.

We furthermore show that CAC is proofwise low over $\mathrm{WKL}_0^{\omega *}$ and therefore that the previous result applies to it. With this we can analyze proofs containing CAC and extract primitive recursive realizers. This is also interesting from the perspective of proof mining, since CAC implies the statement that each sequence of real numbers contains a monotone subsequence, which is commonly used in everyday mathematics.

We start by refining Howard's ordinal analysis of the bar recursor $B_{0,1}$, see [14]. The bar recursor $B_{0,1}$ solves the functional interpretation of Π_1^0-CA (and hence —by iteration— of Π_∞^0-CA). More precisely, an instance of Π_1^0-CA has at most the effect on the growth of functions as an application of $B_{0,1}$ has. Howard's ordinal analysis shows for instance that an application of $B_{0,1}$ to primitive recursive terms (in the sense of Kleene) yields only functions in T_1 (i.e. of Ackermann type). This corresponds to the fact that with Σ_1^0-IA and an instance of Π_1^0-CA one can prove each instance of Σ_2^0-IA and hence the totality of Ackermann function but not the totality of any function on a higher level of the fast growing hierarchy (e.g. functions provably total with Σ_3^0-IA but not with Σ_2^0-IA).

We show that applications of $B_{0,1}$ to terms in $\mathrm{RCA}_0^{\omega *}$ (actually even in $\mathrm{G}_\infty\mathrm{A}^\omega$) yield only primitive recursive functions. Crucial for this analysis is the structure of higher order functionals of $\mathrm{RCA}_0^{\omega *}$. Most important is that this system does not contain a function iterator constant (which in this system is equivalent to Σ_1^0-IA). Our refined ordinal analysis mentioned above corresponds to the fact that QF-IA plus an instance of Π_1^0-CA implies each instance of Σ_1^0-IA and hence the totality of all primitive recursive functions but not of the Ackermann function.

Using this refinement of Howard's ordinal analysis of $B_{0,1}$ we can improve a result from [21] and show that for each principle P which is proofwise low over $\mathrm{WKL}_0^{\omega *}$ the system $\mathrm{WKL}_0^\omega + \Pi_1^0\text{-CP} + \mathrm{P}$ is Π_3^0-conservative over RCA_0^ω and that one can extract primitive recursive realizing terms.

We apply this results to CAC, which lies strictly in between RT_2^2 and COH + Π_1^0-CP, and show that this principle is Π_3^0-conservative over RCA_0^ω and hence does not lead to more than primitive recursive growth. The proof of the lowness of CAC is based on ideas from Chong, Slaman and Yang. However, we will interpret Π^0_1-CP using Π^0_1-CA and hence are able to eliminate it at the end. Therefore, we do not need any non-standard techniques. More importantly and in contrast to the proof of Chong, Slaman and Yang our proof is finitary in the sense of Hilbert's program.

Compared to their result ours is on the one hand weaker in the sense that we only obtain Π^0_3-conservativity not full Π^1_1-conservativity (strictly speaking we also obtain conservativity for sentences of the form $\forall f\, \exists y\, A(f, y)$, where $f \in \mathbb{N}^{\mathbb{N}}$ and $y \in \mathbb{N}$ and A quantifier free). On the other hand our result is stronger since it, additionally, allows term extraction and the simultaneous treatment of WKL. Conservativity for Π^0_3 sentences is optimal for our approach since we eliminate Π^0_1-CP and there are Σ^0_3 consequences of Π^0_1-CP which are not provable in RCA_0, see [1]. Moreover, our conservativity is obtained over a system containing all primitive recursive functionals (in the sense of Kleene) and hence many more statement than in RCA_0 are quantifier free.

The paper is organized as follows. First we give a brief introduction into the logical systems we use. In Section 1 we refine Howard's ordinal analysis of bar recursion. In Section 2 we use this result to refine our techniques from [21] and in Section 3 we show that CAC is proofwise low over a suitable system not containing Σ^0_1-induction and conclude that CAC is Π^0_3-conservative over RCA_0^ω. In the appendix we discus the Erdős-Moser principle. This principle is the counterpart to CAC in the sense that RT^2_2 splits into those two principles.

Logical systems. We will work in fragments of Heyting and Peano arithmetic in all finite types $\mathbf{T}$. The set of all finite types is defined to be the smallest set that satisfies

$$0 \in \mathbf{T}, \qquad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

The type 0 denotes the type of natural numbers and the type $\tau(\rho)$ denotes the type of functions from ρ to τ. The type $0(0)$ is abbreviated by 1 the type $0(0(0))$ by 2. The degree of a type is defined by

$$deg(0) := 0, \qquad deg(\tau(\rho)) := \max(deg(\tau), deg(\rho) + 1).$$

The type of a variable will sometimes be written as superscript.

The systems RCA_0^ω, $\mathrm{RCA}_0^{\omega *}$ are the extensions of RCA_0 resp. RCA_0^* to all finite types. For a detailed definition see [18].

The Grzegorczyk arithmetic in all finite types $\mathrm{G}_\infty\mathrm{A}^\omega$ is defined to be the system that includes λ-abstraction, each branch of the Ackermann function (but not the Ackermann function), bounded search, bounded recursion and quantifier-free induction. Since this system contains each branch of the Ackermann function it contains every primitive recursive function but it does not contain unbounded primitive recursion itself nor unbounded recursors (and hence no function iterator). The closed terms of $\mathrm{G}_\infty\mathrm{A}^\omega$ will be called $\mathrm{G}_\infty\mathrm{R}^\omega$.

The system $\widehat{\mathrm{WE\text{-}PA}}^\omega{\restriction}$ is equivalent to $\mathrm{G}_\infty\mathrm{A}^\omega$ plus Σ^0_1-IA and primitive recursion (of type 0), for a detailed definition see for instance [19, Section 3]. The systems $\widehat{\mathrm{WE\text{-}HA}}^\omega{\restriction}$, $\mathrm{G}_\infty\mathrm{A}^\omega_{\mathrm{i}}$ are the intuitionistic counterparts.

Note that $\widehat{\mathrm{WE\text{-}PA}}^\omega{\restriction}$ and $\mathrm{G}_\infty\mathrm{A}^\omega$ do not satisfy full extensionality. The different variants of extensionality are important in [21] and in the extension of the results from there in Section 2 of this paper. We do not discuss them here and refer the reader to [21, Section 2]. These systems do not satisfy the deduction theorem (this is a consequence of the restricted form for extensionality used). To indicate that a axioms is an implicative assumption we use $\oplus$, e.g. $\mathrm{G}_\infty\mathrm{A}^\omega \oplus \mathrm{WKL} \vdash A$ means $\mathrm{G}_\infty\mathrm{A}^\omega \vdash \mathrm{WKL} \to A$.

Let QF-AC be the schema

$$\forall x\, \exists y\, A_{qf}(x,y) \to \exists f\, \forall x\, A_{qf}(x, f(x)).$$

RCA_0^ω can be embedded into $\widehat{\mathrm{WE\text{-}PA}}^\omega\!\restriction + \text{QF-AC}$ and $\mathrm{RCA}_0^{\omega*}$ can be embedded into $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC}$. The systems with weak König's lemma WKL_0^ω and $\mathrm{WKL}_0^{\omega*}$ can be embedded into $\widehat{\mathrm{WE\text{-}PA}}^\omega\!\restriction + \text{QF-AC} \oplus \mathrm{WKL}$ resp. $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \oplus \mathrm{WKL}$. (Strictly speaking one has to eliminate the extensionality first, see for instance [19, Section 10.4].)

A functional φ is *provably continuous* if there exists a function α_φ such that

$$\forall f\, \exists n\, \alpha_\varphi(\bar{f}n) \neq 0,$$
$$\forall f\, \forall n\, \left(\alpha_\varphi(\bar{f}n) \neq 0 \to \varphi(f) = \alpha_\varphi(\bar{f}n) \dot{-} 1\right).$$

The function α_φ is called *associate*. All closed terms in the system used in this paper are provably continuous, see for instance [19, Proposition 3.57].

1. Ordinal analysis of bar recursion of terms in $\mathrm{G}_\infty\mathrm{R}^\omega$

The goal of this section is to show that a single application of the bar recursor $B_{0,1}$ to terms in $\mathrm{G}_\infty\mathrm{R}^\omega$ does only lead to primitive recursive terms (in the sense of Kleene), i.e. terms with computational size $< \omega^\omega$. We use here the definition of computational size from Howard, see [13, 14]. Roughly speaking the computational size of a term t of type 0 is an upper bound on the number of term reductions one has to apply to obtain a numeral. The computational size of a higher type term t is defined to be the computational size of $t(H_0, \dots, H_n)$ where H_i are fresh variables such that the term is of type 0. Like Howard we assume that a term t has $deg(t) \leq 2$ and is semi-closed (i.e. contains only variables of degree 1 free) whenever we speak about the computational size of a term t.

Recall that the bar recursor $B_{0,1}$ is defined to be

$$B_{0,1}AFGc :=_1 \begin{cases} Gc & \text{if } A[c] < \mathrm{lth}\, c, \\ Fc(\lambda u^0.B_{0,1}(AFG(c * \langle u \rangle))) & \text{otherwise,} \end{cases}$$

where $[c] := \lambda i.(c)_i$.

Howard uses for technical reasons an extension of the term system. This extension is conservative and hence does not lead to any problems. Since we are only going to modify his analysis we will follow this approach:

For each type 1 variable α and terms c, t of type 0 add a new term $\{\alpha, c, t\}$ to the system. The term $\{\alpha, c, t\}$ has the same type as $B_{0,1}A$. The subterms of it consist only of the subterms of t. The purpose of this extension is to bind all occurrences of α in t. The term $B_{0,1}AFGc$ is equal to $\{\alpha, c, A\alpha\}FGc$ and can also be contracted to this term. The term $\{\alpha, c, t\}$ satisfies following contractions:

$$\begin{array}{rcll} \{\alpha, c, t\} & \textbf{contr} & \{\alpha, c, t'\} & \text{if } t \textbf{ contr } t' \\ \{\alpha, c, i\}FGc & \textbf{contr} & Gc & \text{if } i \text{ is numeral} < \mathrm{lth}(c) \\ \{\alpha, c, t\}FGc & \textbf{contr} & M & \\ \{\alpha, c, t\}FG(c * \langle n \rangle) & \textbf{contr} & \{\alpha, c * \langle n \rangle, t\}FG(c * \langle n \rangle) & \end{array}$$

where

$$M := \begin{cases} Gc & \text{if } t[\lambda i.(c)_i/\alpha] < \mathrm{lth}(c), \\ Fc\big(\lambda u.\{\alpha, c, t\}FG(c * \langle u\rangle)\big) & \text{otherwise.} \end{cases} \tag{3}$$

For details we refer the reader to [14]. Note that $\{\alpha, c, t\}$ is there defined for bar recursors of arbitrary types and not only for $B_{0,1}$.

We now state a modified version of Theorem 2.3 of [14]. The proof of the following theorem differs from Howard's proof only in using other ordinal estimates. The result of it is more suitable for terms which have finite computational size because it shows in this case that the resulting term has computational size $< \omega^\omega$, whereas in Howard's theorem the computational size is always $\geq \omega^\omega$. For parameters which have computational size of an infinite ordinal Howard's theorem yields better results.

Theorem 1. *Let F, G and t have computational sizes f, g and $\mathit{size}(t)$. Then the term $\{\alpha, c, t\}FGc$ has computational size 2^{g+f4h}, where $h = \omega + \omega \mathit{size}(t) + \omega$.*

Proof. We assume that $f, g \geq 1$.

Like Howard, we say for a term $\{\alpha, d, s\}$ that the sequence d is m-critical in s if the term to be contracted in s is of the form αm and $m \geq \mathrm{lth}(d)$. We define $\mathit{ord}(\alpha, d, s)$ to be $\omega + \omega \mathit{size}(s) + 1$ if d is not critical in s and s is not a numeral. If d is m-critical we let $\mathit{ord}(\alpha, d, s) = \omega + \omega \mathit{size}(s) + m - \mathrm{lth}(d) + 3$. If s is a numeral n, we let $\mathit{ord}(\alpha, d, s) = \omega + (n \dot{-} \mathrm{lth}(d)) + 2$.

Like in [14, Theorem 2.3] we prove by transfinite induction on $b = \mathit{ord}(\alpha, c, t)$ that $\{\alpha, c, t\}FGc$ has computational size 2^{g+f4b}.

We consider the following cases:

- If t is not a numeral and c is not critical then executing a computation step reduces t to t' such that $\mathit{size}(t') < \mathit{size}(t)$ and hence $\mathit{ord}(\alpha, c, t') < \mathit{ord}(\alpha, c, t)$ and so $2^{g+f4\,\mathit{ord}(\alpha,c,t')} < 2^{g+f4b}$.
- If t is a numeral that is $< \mathrm{lth}(c)$ then $\{\alpha, c, t\}FGc$ reduces to Gc which has computation size $g \leq 2^g < 2^{g+f4b}$.
- The cases where c is critical or t is a numeral $\geq \mathrm{lth}(c)$ remain. We treat here at first the former case, the later will follow from a slight modification of this.

  We can reduce $\{\alpha, c, t\}FGc$ to M from (3) in one step. For the case distinction in M we have to compute $t[\lambda i.(c)_i/\alpha]$. By Theorem 2.1 from [14] we can compute it in $\omega \mathit{size}(t)$ steps. By finitely many steps j we then arrive at either

$$Gc \qquad \text{or} \qquad \underbrace{Fc\big(\lambda u.\{\alpha, c, t\}FG(c * \langle u\rangle)\big)}_{M_2}.$$

  In the case of Gc additionally g more computation steps are needed. In total this yields

$$g + \underbrace{j + \omega \mathit{size}(t) + 1}_{<b} < 2^{g+f4b}. \tag{4}$$

  In the case of M_2 we reduce

$$\lambda u.\{\alpha, c, t\}FG(c * \langle u\rangle)x \qquad \text{to} \qquad \{\alpha, c * \langle n\rangle, t\}FG(c * \langle n\rangle)$$

in 3 steps. Let $a = ord(\alpha, c * \langle n \rangle, t)$. By definition of ord we have $a < b$. By induction hypothesis $\{\alpha, c * \langle n \rangle, t\} FG(c * \langle n \rangle)$ has computational size 2^{g+f4a}. The term c has computational size $\omega \leq 2^{g+f4a}$. Together with Theorem 2.1 from [14] this show that M_2 has computation size

$$\begin{aligned}
(2^{g+f4a}+3)f &\leq (2^{g+f4a}+2^{g+f4a})f && (a \geq \omega)\\
&\leq 2^{g+f4a+1} \cdot f\\
&< 2^{g+f4a+1} \cdot 2^{f+1} && (f < 2^{f+1})\\
&= 2^{g+f4a+1+f+1}\\
&\leq 2^{g+f4a+f3} && (f \geq 1)
\end{aligned}$$

Together with the steps for the cases distinction we obtain the following computational size

$$\begin{aligned}
(2^{g+f4a}+3)f + \underbrace{j + \omega size(t) + 1}_{=:z} &< 2^{g+f4a+f3} + 2^{z+1}\\
&\leq 2^{\max(g+f4a+f3, z+1)} \cdot 2\\
&\leq 2^{g+f4b}
\end{aligned}$$

The last $\leq$ holds since $\max(g + f4a + f3, z + 1) < g + f4b$ and therefore $\max(g + f4a + f4, z + 1) + 1 \leq g + f4b$.

The case where t is a numeral $\geq \mathrm{lth}(c)$ can be treated similarly. Here $t[\lambda i.(c)_i/\alpha]$ does not need to be computed. Hence, the equation (4) becomes

$$g + j + 1 < 2^{g+f4b}.$$

Since $j + 1 < \omega < b$ this is still valid. The rest of the argument remains the same because also $a < b$ holds.

This proves the theorem. □

Remark 2. Define Bezem's bar recursor $B^B_{0,1}$ to be

$$B^B_{0,1} AFGc :=_1 \begin{cases} Gc & \text{if } A[c]^B < \mathrm{lth}\, c,\\ Fc(\lambda u^0.B^B_{0,1}(AFG(c * \langle u \rangle))) & \text{otherwise,} \end{cases}$$

where $[c]^B := \begin{cases} (c)_i & \text{if } i < \mathrm{lth}(c)\\ (c)_{\mathrm{lth}(c) \dot{-} 1} & \text{otherwise.} \end{cases}$

This bar recursor differs from Howard's bar recursor only in the definition of $[\cdot]$. Hence, Theorem 1 also holds for $B^B_{0,1}$.

We will use this bar recursor in Theorem 5 below to define a majorant for $B_{0,1}$.

In the following we will treat $B^{(B)}_{0,1}$ as a constant satisfying the defining equations of the bar recursor, but which is *not* provably total.

Theorem 3. *The system* $\widehat{\text{WE-PA}}^\omega\!\restriction$ *proves that for all semi-closed terms* A, F, G, c *with provably finite computational size* $B_{0,1}AFGc$ *is total, i.e. there exists a term that provably satisfies the defining equations. The same holds for* $B^B_{0,1}AFGc$.

Proof. Let f, g, a be the computational sizes of F, G, A.

The proof of Theorem 1 for $\{\alpha, c, A\alpha\}FGc$ can be formalized in a system containing the Σ^0_1-least number principle for sets containing elements $< 2^{g+f4(\omega+\omega a+\omega)}$. Since

$$2^{g+f4(\omega+\omega a+\omega)} = 2^{\omega(a+2)} = \omega^{a+2} < \omega^{\omega}$$

this principle is equivalent to Σ^0_1-induction (over $\mathbb{N}$), see [10, II.3.18] and also Theorem 56 in [21]. Hence the system $\widehat{\text{WE-PA}}^{\omega}\!\restriction$ suffices.

The conservativity of Howard's extended term system can also be formalized in $\widehat{\text{WE-PA}}^{\omega}\!\restriction$. Therefore this systems also proves the totality of $B_{0,1}AFGc$. □

For the analysis of terms in $\mathrm{G}_\infty\mathrm{R}^\omega$ we use the following property:

Proposition 4 ([16, Proposition 2.2.22], [19, Corollary 3.42])**.** *Let $\rho = 0\rho_k \ldots \rho_1$ with $deg(\rho_i) \leq 1$. For each term $t^\rho \in \mathrm{G}_\infty\mathrm{R}^\omega$ there exists a term $t^*[x_1^{\rho_1}, \ldots, x_k^{\rho_k}]$ such that*

- *$t^*[x_1, \ldots, x_k]$ contains at most $x_1, \ldots, x_k$ as free variables,*
- *$t^*[x_1, \ldots, x_k]$ is build up only from $x_1, \ldots, x_k, 0^0, A_0, A_1, \ldots,$ where A_i is the i-th branch of the Ackermann function,*
- $\mathrm{G}_\infty\mathrm{A}^\omega_{\mathrm{i}} \vdash \lambda x_1, \ldots, x_k.t^*[x_1, \ldots, x_k] \text{ maj } t$.

In particular, every term $t \in \mathrm{G}_\infty\mathrm{R}^\omega$ of degree ≤ 2 is provably majorized by a term that has provably finite computational size.

Theorem 5. *Let $A[x^1], F[x], G[x], c[x]$ be terms of appropriated type such that $B_{0,1}AFGc$ is well-formed and such that $\lambda x^1.A[x], F[x], G[x], c[x] \in \mathrm{G}_\infty\mathrm{R}^\omega$. Then $\widehat{\text{WE-HA}}^{\omega}\!\restriction$ proves that $f := \lambda x^1.\lambda y^0.B_{0,1}AFGcy$ is total. Moreover this system proves that there exists a majorant to f.*

Proof. First observe that the totality of the bar recursor in f can be proven using Π^0_2-bar induction of type 0 (Π^0_2-BI_0). (Use the bar induction to prove the statement $\forall u\, \exists v\, B_{0,1}AFGcu = v$. For a definition of BI_0 see for instance [21, Definition 52].) To make use of the properties described in Proposition 4 we will first show that a majorant to f exists. With this we can bound the $\exists$-quantifier in the bar induction and obtain that Π^0_1-bar induction (Π^0_1-BI_0) suffices. By Lemma 53 in [21] this is included in $\widehat{\text{WE-PA}}^{\omega}\!\restriction + \text{QF-AC}$.

We now show that there exists majorant to f and that it is total. Let

$$B^\times_{0,1} := \lambda A, F, G, c.B^B_{0,1}AF_GGc\,,$$

$$B^*_{0,1} := \lambda A, F, G, c.(B^\times_{0,1}AFGc)^M, \tag{5}$$

where

$$F_Gtf := \max(Gt, Ftf_{(\mathrm{lth}(t)\,\dot{-}\,1)}), \qquad f_i(x) := f(\max(i, x))$$
$$\text{and} \qquad (f)^M x := \max_{y \leq x} f(x).$$

We have $B^*_{0,1}$ maj $B_{0,1}$ provably in $\widehat{\text{WE-PA}}^{\omega}\!\restriction + \text{QF-AC}$, see Proposition 54 in [21] and also [2]. In [21, Proposition 54] we use a different majorant but mutatis mutandis the proof also shows that $B^*_{0,1}$ as defined in (5) majorizes $B_{0,1}$.[1]

[1] We do not use here the majorant of $B_{0,1}$ as defined in [19] or [21] which would build internally paths through the tree A which are *not* monotone. Before applying the majorant A^* to such paths they have to be made monotone such that they are majorants. But this cannot be done using only terms with finite computational size.

Applying Proposition 4 we obtain majorizing semi-closed terms A^*, F^*, G^*, c^* for A, F_G, G, c with finite computational size.

Since $B^*_{0,1}$ is a specific application of $B^B_{0,1}$, we can apply Theorem 3 to $B^*_{0,1}A^*F^*G^*c^*$ to obtain its totality. With this the totality of f and the existence of a majorant is proven in the system $\widehat{\text{WE-PA}}^\omega\!\upharpoonright + \text{QF-AC}$.

Since this statement is $\forall\exists$, the functional translates this prove into a proof in $\widehat{\text{WE-HA}}^\omega\!\upharpoonright$. This provides the theorem. ☐

Corollary 6. *The term $B_{0,1}AFGc$ where A, F, G, c are semi-closed terms of* $\text{G}_\infty\text{A}^\omega$ *is provably equal to a term in T_0 (i.e. the fragment of Gödel's T where the recursor is restricted to recursion of type* 0*).*

Proof. Apply the functional interpretation (combined with a negative translation) to the result of Theorem 5, see [19, Proposition 10.53]. The term extracted using this satisfies the corollary. ☐

This result can be used to reprove the following result from Parsons [23, Lemma 4].

Corollary 7. *Let R_1 be the recursor for type* 1 *objects, i.e. $R_1 0 f G x = fx$ and $R_1(n+1)fGx = G(R_1 n f G)nx$, where x, n, fx are of type* 0*. (Note that R_1 cannot be reduced to primitive recursion, since G takes an element of* $\mathbb{N}^\mathbb{N}$ *as first parameter.)*

Then the term $R_1 n f G$ where G is a semi-closed term of $\text{G}_\infty\text{A}^\omega$ *is provably equal to a term in T_0.*

Proof. Corollary 6 and the fact that R_1 is elementarily definable from $B_{0,1}$. ☐

2. Proofwise low relative to $\text{G}_\infty\text{A}^\omega$

In [21] we showed that principles P of the form

$$(\text{P})\colon \forall c^1\, \exists g^1\, \underbrace{\forall u^1\, P_{qf}(c,g,u)}_{\equiv: P(c,g)}, \tag{6}$$

where P_{qf} is quantifier free, which are proofwise low relative to $\widehat{\text{WE-PA}}^\omega\!\upharpoonright + \text{QF-AC} \oplus \text{WKL}$ are conservative over $\widehat{\text{WE-PA}}^\omega\!\upharpoonright + \Sigma^0_2\text{-IA}$ for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x,y)$.

We now show that for principles P which are proofwise low relative to $\text{G}_\infty\text{A}^\omega + \text{QF-AC} \oplus \text{WKL}$ the system $\widehat{\text{WE-PA}}^\omega\!\upharpoonright + \text{QF-AC} \oplus \text{WKL} \oplus \text{P}$ is conservative over $\widehat{\text{WE-HA}}^\omega\!\upharpoonright$ for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x,y)$. (Actually we only treated the case of RT^2_2 but mutatis mutandis this works for each principle of this form.) For notation and a discussion of the techniques involved in this proof we refer the reader to [21].

Let now P be a principle that is proofwise low over $\text{G}_\infty\text{A}^\omega + \text{QF-AC} \oplus \text{WKL}$ (a fortiori it is sufficient that P is proofwise low over $\text{WKL}_0^{\omega *}$ since this system can be embedded into the other). This means we have for each provably continuous term φ a provably continuous term ξ such that

$$\text{G}_\infty\text{A}^\omega + \text{QF-AC} \oplus \text{WKL} \vdash \forall c\, \left(\Pi^0_1\text{-CA}(\xi c) \to \exists g\, \left(P(c,g) \wedge \Pi^0_1\text{-CA}(\varphi c g)\right)\right).$$

A functional interpretation of this statement yields

$$\text{(7)}\quad \mathrm{G}_\infty\mathrm{A}^\omega_\mathrm{i} \oplus (\mathcal{B}) \vdash$$
$$\forall c\, \forall U\, \forall f_\xi\, \forall X_\varphi, Y_\varphi\, \exists x_\xi, y_\xi\, \exists g\, \exists f_\varphi\, \Big(\big(\Pi^0_1\text{-}\widehat{\mathrm{CA}}(\xi f)\big)_{qf}(f_\xi, x_\xi, y_\xi)$$
$$\to \big(P(c,g,Ugf_\varphi) \wedge \Pi^0_1\text{-}\widehat{\mathrm{CA}}(\varphi f g)\big)_{qf}(f_\varphi, X_\varphi g f_\varphi, Y_\varphi g f_\varphi))\Big),$$

and that there exist terms in $\mathrm{G}_\infty\mathrm{R}^\omega$ realizing $x_\xi, y_\xi, g, f_\varphi$, cf. to Theorem 51 in [21].

Using (7) in the proof of Proposition 61 from [21] instead of Theorem 51 of [21] we obtain a variant of Proposition 61 where $\widehat{\text{WE-HA}}^\omega\!\restriction$ is replaced by $\mathrm{G}_\infty\mathrm{A}^\omega_\mathrm{i}$, RT^2_2 is replaced by P and $T_0[\mathcal{R}]$ is replaced by $\mathrm{G}_\infty\mathrm{R}^\omega[\mathcal{R}]$ (here $\mathcal{R}$ is now a solution functional for P^{ND}). In the same way we obtained Corollary 62 from Proposition 61 in [21] we can extend the previous statement to terms in $\mathrm{G}_\infty\mathrm{R}^\omega[\mathcal{R}, R_0, \Phi_0']$ (which is equal to $T_0[\mathcal{R}, \Phi_0']$) but of course *not* to terms containing R_1. As consequence we obtain the following modification of Proposition 63 from [21]:

Proposition 8. *Let A_{qf} be a quantifier-free formula that contains only the shown variables free and let* P *be a principle of the form* (6) *which is proofwise low over* $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \oplus \text{WKL}$. *If*

$$\widehat{\text{N-PA}}^\omega\!\restriction + \text{QF-AC} + \text{WKL} + \mathrm{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

then one can find terms $t_y, t_u, t_v, \xi \in \mathrm{G}_\infty\mathrm{R}^\omega[\mathcal{B}]$ such that

$$\mathrm{G}_\infty\mathrm{A}^\omega_\mathrm{i} \oplus \mathcal{B} \vdash \forall x^1\, \forall f\, \Big(\big(\Pi^0_1\text{-}\widehat{\mathrm{CA}}(\xi x)\big)_{QF}(f, t_u f x, t_v f x) \to A_{qf}(x, t_y f x)\Big).$$

Similarly to the discussion preceding Theorem 65 in [21], we interpret $\Pi^0_1\text{-}\widehat{\mathrm{CA}}(\xi x)$ with a single application of $B_{0,1}$ (or in other words using a single application of the rule of bar recursion). With this we obtain

$$\widehat{\text{WE-HA}}^\omega\!\restriction \oplus (\mathcal{B}) + \text{R-}(B_{0,1}) \vdash \forall x^1\, A_{qf}(x, tx),$$

where $t \in \mathrm{G}_\infty\mathrm{R}^\omega[\mathcal{B}, B_{0,1}]$ and t contains only a single application of $B_{0,1}$ to semi-closed terms $A[x], F[x], G[x], c[x]$ and R-$(B_{0,1})$ is the rule of $B_{0,1}$ which states that applications of $B_{0,1}$ to semi-closed term of $\mathrm{G}_\infty\mathrm{R}^\omega$ exists. We strengthened the verifying theory to $\widehat{\text{WE-HA}}^\omega\!\restriction$ because we do not know whether one can show without Σ^0_1-IA that an application of $B_{0,1}$ solves the functional interpretation of an instance of Π^0_1-CA.

We now build a majorant t^* of t. The application of $B_{0,1}$ will be majorized like in the proof of Theorem 5. By Proposition 54 in [21] and the fact that the the theory used in this Proposition has a functional interpretation in $\widehat{\text{WE-HA}}^\omega\!\restriction$, we obtain that $B^*_{0,1}$ applied to majorants of A, F, G, c majorizes $B_{0,1}AFGc$ Hence we obtain

$$\widehat{\text{WE-HA}}^\omega\!\restriction \oplus (\mathcal{B}) + \text{R-}(B_{0,1}) \vdash \forall x^1\, \exists y \le t^* x\ A_{qf}(x,y),$$

where $t^* \in \mathrm{G}_\infty\mathrm{R}^\omega[B_{0,1}]$ and t^* contains only a single application of $B_{0,1}$ to semi-closed terms with finite computational size.

Applying bounded search we obtain a new realizer t' for y:

$$t'x := \begin{cases} \text{minimal } y \le t^* x \text{ with } A_{qf}(x,y), & \text{if such a } y \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Now using the ordinal analysis of $B_{0,1}$ we obtain a term t'' that is provably equal to t' and that is definable using transfinite primitive recursion up to $< \omega^\omega$ and hence in $\widehat{\text{WE-HA}}^\omega\!\upharpoonright$, see [10, II.3.18] and also [21, Theorem 56]. So that

$$\widehat{\text{WE-HA}}^\omega\!\upharpoonright \oplus (\mathcal{B}) \vdash \forall x^1\, A_{qf}(x, t''x).$$

The principle $\mathcal{B}$ may be eliminate from the system with a monotone functional interpretation like in [21], see [15], [19, Section 10.3]. We obtain

$$\widehat{\text{WE-HA}}^\omega\!\upharpoonright \vdash \forall x^1\, A_{qf}(x, t''x).$$

Combining this discussion with Proposition 8 we obtain the following theorem:

Theorem 9. *Let $A_{qf}(x^1, y^0)$ be a quantifier-free formula with only x, y free and* P *a principle of the form* (6) *which is proofwise low over* $\text{G}_\infty\text{A}^\omega + \text{QF-AC} \oplus \text{WKL}$. *If*

$$\widehat{\text{N-PA}}^\omega\!\upharpoonright + \text{QF-AC} + \text{WKL} + \text{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

then one can extract a term $t \in T_0$ such that

$$\widehat{\text{WE-HA}}^\omega\!\upharpoonright \vdash \forall x^1\, A_{qf}(x, tx).$$

Together with elimination of extensionality (see [22], [19, Section 10.4] and also [21, Proposition 7]) we obtain:

Corollary 10. *If*

$$\widehat{\text{E-PA}}^\omega\!\upharpoonright + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} + \text{WKL} + \text{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

then one can extract a term $t \in T_0$ such that

$$\widehat{\text{WE-HA}}^\omega\!\upharpoonright \vdash \forall x^1\, A_{qf}(x, tx).$$

Corollary 11. *Let* P *be a principle of the form* (6) *that is proofwise low over* $\text{WKL}_0^{\omega *}$. *Then the system* $\text{WKL}_0^\omega + \text{P}$ *is conservative over* RCA_0^ω *for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x, y)$. Moreover, one can extract from a proof of this statement a term $t \in T_0$ realizing y (that is a primitive recursive functional in the sense of Kleene).*

In particular, $\text{WKL}_0^\omega + \text{P}$ *is Π^0_3-conservative over* RCA_0^ω *and Π^0_2-conservative over* PRA.

Proof. The first part of this corollary is just a reformulation of the previous corollary. The second part follows from the observation that over RCA_0^ω each Π^0_3-sentence is equivalent to a sentence of the form $\forall x^1\, \exists y^0\, A_{qf}(x, y)$. The last statement follows from the fact that RCA_0^ω is Π^0_2-conservative over PRA. □

3. Chain antichain principle

Let the chain antichain principle (CAC) be the principle that states that every partial order on $\mathbb{N}$ has an infinite chain or antichain. For notational ease we assume here that each (anti)chain is also ordered by the ordering of $\mathbb{N}$. We formalize CAC in the following way:

$$\begin{aligned}(\text{CAC})\colon \forall \chi_P\, \exists H \Big(\quad & \forall u, v \in H\, \big(u < v \to u \leq_P v\big) \\ \vee\ & \forall u, v \in H\, \big(u < v \to u \geq_P v\big) \\ \vee\ & \forall u, v \in H\, \big(u < v \to u \mid_P v\big)\Big),\end{aligned}$$

where the set H is a given as strictly increasing enumeration, i.e. H is a function such that Hn is the n-th element of H.[2] The partial order P is given by its characteristic function χ_P. The relations $\leq_P$, $|_P$ are defined to be

$$u \leq_P v :\equiv \begin{cases} \chi_P(x,y) = 0 & \begin{array}{l}\text{The relation } ([0,\langle u,v\rangle], \preceq) \text{ with} \\ x \preceq y :\equiv [\langle x,y\rangle \leq \langle u,v\rangle \wedge \chi_P(x,y) = 0] \\ \text{defines a partial order,}\end{array} \\ \bot & \text{otherwise,} \end{cases}$$

$$u \,|_P\, v :\equiv \neg\,(u \leq_P v) \wedge \neg\,(v \leq_P u)\,.$$

(We assume here that the paring $\langle x,y\rangle$ is monotone in both components.) With this any function χ_P describes a partial order.

Hirschfeldt and Shore observed in [11] that CAC splits into the cohesive principle and the, so called, stable chain antichain principle. The *cohesive principle* (COH) is the statement that for each sequence $(R_i)_{i\in\mathbb{N}}$ if subsets of $\mathbb{N}$ there exists a cohesive set X, i.e. a set X satisfying

$$\forall i\ \left(X \subseteq^* R_i \wedge X \subseteq^* \overline{R_i}\right),$$

where $X \subseteq^* Y :\equiv (X \setminus Y \text{ is finite})$. The *stable chain antichain principle* (SCAC) is the restriction of CAC to stable partial ordering, where we call a partial ordering $\leq_P$ *stable* if one of the following holds

(i) For all x either $x \leq_P y$ for all but finitely many y or $x \,|_P\, y$ for all but finitely many y.
(ii) For all x either $x \geq_P y$ for all but finitely many y or $x \,|_P\, y$ for all but finitely many y.

Remark 12. In [20] we showed that $\text{COH} + \Pi^0_1\text{-CP}$ is equivalent to the variant of the Bolzano-Weierstraß principle that states that every bounded sequence of $\mathbb{R}$ has a —possibly slowly— converging subsequence.

The principle ADS, which is CAC restricted to linear orders, is equivalent to the statement that every sequence in $\mathbb{R}$ has a monotone subsequence. If the sequence is bounded then the monotone subsequence is a fortiori converging (possible slowly). Hence ADS and CAC can be seen as generalizations of this variant of the Bolzano-Weierstraß principle.

To see that ADS implies that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ has an monotone subsequence one has take some care since equality on $\mathbb{R}$ and hence also $\leq_\mathbb{R}$ is not decidable. To prove the statement one has to make the following case distinction. Either (x_n) has a constant subsequence or there exists a subsequence of pairwise different elements. The solution to the former case is trivial and the latter case can be solved by applying ADS since $\leq_\mathbb{R}$ coincides with $<_\mathbb{R}$ on this sequence and is therefore decidable.

For the other direction it suffices to show that each countable linear ordering can be embedded into a subset of $\mathbb{Q}$. This follows from the construction described in the proof of [8, Theorem 2.1] and by noting that it can be carried out in RCA_0.

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[2] Strictly speaking we cannot quantify over strictly monotone functions. Officially, we quantify over all functions from $\mathbb{N} \to \mathbb{N}$ and replace every occurrence of $H(n)$ by

$$\tilde{H}(n) := \begin{cases} H(n) & \text{if } n = 0 \text{ or } H(n) > \tilde{H}(n \dot{-} 1), \\ \tilde{H}(n \dot{-} 1) + 1 & \text{otherwise.} \end{cases}$$

Here it is also interesting to mention that de Smet and Weiermann did a fine grain analysis of a density variant of this principle restricted to natural numbers in [7, 6].

We will show in this section that CAC is proofwise low over $G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL}$ and hence that Theorem 9 and the Corollaries 10 and 11 apply to it. This strengthens our result from [21], where we were only able to handle COH.

Our proof is based on [5]. The non-standard construction is replaced by the following argument.

3.1. **Building infinite sets without Σ^0_1-induction.** Call a set X

- *infinite* or *unbounded* if

$$\forall k \, \exists n > k \, n \in X$$

and

- *strictly increasingly enumerable* if there

exists a strictly monotone function f such that $rng(f) = X$.

It is clear that a strictly increasingly enumerable set is also unbounded. However, to construct a strictly increasing enumeration for an unbounded set in general requires Σ^0_1-IA (e.g. RCA_0 or $\widehat{\text{WE-HA}}^\omega\restriction + \text{QF-AC}$).

We will now discuss a way to build unbounded sets in a system that does not contain Σ^0_1-IA. Let f be a function that maps (codes of) finite subsets of $\mathbb{N}$ into (codes of) finite subsets of $\mathbb{N}$ and that is monotone in the sense of

$$x \subsetneq f(x), \qquad f(x) \setminus x \subseteq [\max(x)+1, \infty[\,. \tag{8}$$

Define now $X \subseteq \mathbb{N}$ by

$$X := \bigcup_{n \in \mathbb{N}} f^n(\emptyset),$$

where f^n is the n-th iteration of f.

The properties of f ensure that

$$n \in X \quad \longleftrightarrow \quad n \in f^{n+1}(\emptyset).$$

Hence, the function $g(n) := \left[n\text{-th element of } f^{n+1}(\emptyset)\right]$ defines a strictly increasing enumeration of X that is definable for instance in RCA_0 or $\widehat{\text{WE-HA}}^\omega\restriction + \text{QF-AC}$ (if f is).

In a system without Σ^0_1-IA (e.g. RCA^*_0 or $G_\infty A^\omega + \text{QF-AC}$) it is a priori not clear whether X is well defined since one cannot build the n-th iterate of the unbounded function f.

To define a set that is provably equal to X let

$$\tilde{f}_k(x) := \begin{cases} f(x) & \text{if } f(x) \subseteq [0,k[, \\ x & \text{otherwise.} \end{cases}$$

The function $\tilde{f}_k$ is bounded and therefore can be iterated using bounded recursion. For $\tilde{f}_k$ we have the following equivalence

$$n \in X \quad \longleftrightarrow \quad n \in f^{n+1}(\emptyset) \quad \longleftrightarrow \quad n \in f\left((\tilde{f}_n)^n(\emptyset)\right).$$

To see that the last equivalence holds let m' be the least $m \leq n+1$ with $f^m(\emptyset) \cap [n,\infty[\neq \emptyset$. By (8) we have $f^{(m' \dot{-} 1)}(\emptyset) \subseteq [0,n[$ and hence $(\tilde{f}_n)^n(\emptyset) = f^{(m' \dot{-} 1)}(\emptyset)$ and $f(\tilde{f}_n)^n(\emptyset) = f^{m'}(\emptyset)$.

Therefore, we can define that characteristic function χ_X by

$$\chi_X(n) := \begin{cases} 0 & \text{if } n \in f\left((\tilde{f}_n)^n(\emptyset)\right), \\ 1 & \text{otherwise.} \end{cases}$$

To show now that X is unbounded assume for a contradiction that X is bounded by b. By the definition of X we then have that $(\tilde{f}_{b+1})^n(\emptyset) = f^n(\emptyset)$. Hence f is also bounded (at least along the iteration). Therefore bounded recursion suffices to iterate the function and the strictly increasing enumeration g of the set X can be defined. But this contradicts the boundedness of X. Hence X is unbounded.

3.2. **Proofwise low.** We will use the ideas of the preceding section to show that CAC is proofwise low over $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \oplus \text{WKL}$. To apply these ideas let uCAC be the CAC with the except that it only require an unbounded (anti)chain, i.e.

$$\begin{aligned}(\text{uCAC})\colon \forall \chi_P\, \exists H = \chi_H, f_H \Big(&\forall n \max(f_H(n), n) \in H \\ &\wedge \Big(\quad \forall u, v \in H \left(u < v \rightarrow u \leq_P v\right) \\ &\quad \vee \forall u, v \in H \left(u < v \rightarrow u \geq_P v\right) \\ &\quad \vee \forall u, v \in H \left(u < v \rightarrow u \mid_P v\right)\Big)\Big).\end{aligned}$$

Here H is given as a characteristic function χ_H plus a witness for the unboundedness f_H (i.e. $f_H(n) \geq n$ and its range is included in H). Let uSCAC be the restriction of uCAC to stable partial orderings.

For a partial order $\leq_P$ define

$$A_\Box := \{x \mid x \,\Box\, y \text{ for all but finitely many } y\},$$

where $\Box \in \{\leq_P, \geq_P, |_P\}$. If $\leq_P$ is stable then these sets are disjoint and either $A_{\leq_P} \cup A_{|_P} = \mathbb{N}$ or $A_{\geq_P} \cup A_{|_P} = \mathbb{N}$. Hence these sets are Δ^0_2. One can easily establish that each infinite chain, antichain is a subset of $A_{\leq_P}$ resp. $A_{\geq_P}$, $A_{|_P}$.

We will write in the following $y \subseteq^{fin} X$ for y is a code for a finite subset of X and $y \sqsubseteq X$ for y is an initial segment of the strictly increasing enumeration of the set X.

Proposition 13. *For every closed term φ there exists a closed term ξ such that*

$$\begin{aligned}&\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \\ &\qquad \vdash \forall \chi_P \left(\Pi^0_1\text{-CA}(\xi\chi_P) \rightarrow \exists H, f_H \left(\text{uSCAC}(\chi_P, H) \wedge \Pi^0_1\text{-CA}(\varphi\chi_P H f_H)\right)\right).\end{aligned}$$

Here uSCAC(χ_P, H, f_H) *expresses that H, f_H is a solution to* uSCAC *and the partial order described by χ_P.*

In other words uSCAC *is proofwise low over* $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC}$.

Proof. Let χ_P be the characteristic function of a stable partial ordering. Without loss of generality we assume that (i) from the definition of stability holds, the case (ii) can be handle analogously.

We will start with the following claim:

Claim: Let Y be an infinite Σ^0_1-set whose characteristic function is given by a term t which contains only χ_P and type 0 variables free. This means $n \in Y$ iff $\exists x\, tnx = 0$. Then Y either has an element in $A_{\leq_P}$ or one can define an infinite antichain that solves the lemma.

Proof of the claim: Suppose that Y does not contain an element of $A_{\leq_P}$ i.e. $Y \subseteq A_{|_P}$. By an instance of Π^0_1-CP (which follows from the instance of Π^0_1-CA) one can proof that

$$\forall y \subseteq^{fin} Y \ (y \text{ is an antichain} \to \exists z \in Y \ y \cup \{z\} \text{ is an antichain}) .$$

By definition this is equivalent to

$$\forall y \, \forall x \ (\forall i < \mathrm{lth}(y) \ t(y)_i(x)_i = 0 \wedge y \text{ is an antichain}) \\ \to \exists z, x' \ (tzx' = 0 \wedge y \cup \{z\} \text{ is an antichain}) .$$

Now let f be the choice function that chooses the minimal z (and x') extending y (and x). Iterating f using an instance of Σ^0_1-IA (which also follows from the instance of Π^0_1-CA) yields an infinite antichain H. The instance of comprehension Π^0_1-CA($\varphi\chi_P H$) can be reduced to the imposed instance of comprehension using the following equivalence

$$\forall n \ (\forall k \, \varphi\chi_P Hnk \leftrightarrow \forall k \, \forall x \, \forall h \sqsubseteq H \ \alpha_{\varphi\chi_P}(h, n, k) \leq 1)$$

and the fact that $h \sqsubseteq H$ can be using a quantifier-free formula depending only on t, x, h. (This formula just expresses that h, x are the result of the iteration of f.) The function $\alpha_{\varphi\chi_P}(h, n, k)$ here is an associate to the function $\lambda H.\varphi\chi_P Hnk$. For notational ease we assume here that H is given as strictly increasing enumeration. Since on can define from this a characteristic function for H and f_H by a term in $\mathrm{G}_\infty\mathrm{A}^\omega$ this does not lead to any problems. This proves the claim.

We assume from now on that there is no Σ^0_1-set $Y \subseteq A_{|_P}$ given by such a term t. Otherwise we would be done. The assumption implies that $A_{\leq_P}$ has infinitely many elements. (If not the set $Y := [\max(A_{\leq_P}) + 1, \infty[$ would be an infinite subset of $A_{|_P}$ which could be easily described by a term.) We will show that we can construct an unbounded $\leq_P$-chain $H \subseteq A_{\leq_P}$ for which we can prove the instance of Π^0_1-CA.

First we define a function $g_1(n, h)$ that for a given n extends a given $\leq_P$-chain $h \subseteq^{fin} A_{\leq_P}$ to a finite $\leq_P$-chain $h' \subseteq^{fin} A_{\leq_P}$ such that for all $\leq_P$-chains X with $h' \sqsubseteq X$ and $X \subseteq A_{\leq_P}$ the following holds

$$\forall n' < n \ (\forall k \ \varphi\chi_P Xn'k = 0 \leftrightarrow \forall k \ \alpha_{\varphi\chi_P}(h', n', k) \leq 1) . \tag{9}$$

In other words we extend h to h' such that the instance of comprehension Π^0_1-CA($\varphi\chi_P H$) is decided up to the index n.

Define for each $D \subseteq [0, n]$ the set

$$S_{D,h} := \{h' \mid h' \text{ is a } \leq_P\text{-chain} \wedge h \sqsubseteq h' \wedge |h'| < \infty \wedge \forall n' \in D \, \exists k \, \alpha_{\varphi\chi_P}(h', n', k) > 1\}.$$

The elements of this set are those extensions of h which make the comprehension Π^0_1-CA($\varphi\chi_P H$) for the indexes in D false. This set is Σ^0_1 and can be defined by a fixed term containing only the parameters χ_P, D, h.

The statement that there is no extension of h in $S_{D,h}$ whose elements are in $A_{\leq_P}$ is

$$\forall y \ \big(y \notin S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\big) . \tag{10}$$

This formula is Π^0_2. We will show that there exists a Σ^0_2 formula that is equivalent and hence that the statement is Δ^0_2.

Consider the set $M_{D,h} := \{\max_P(y) \mid y \in S_{D,h}\}$. This set is also Σ^0_1 also does only depend on χ_P and the type 0 objects D, h. (Recall that we assume that a $\leq_P$-chain is also ordered by $<$ on $\mathbb{N}$.)

We will distinguish the following cases:

- The set $M_{D,h}$ is infinite. In this case there exists by the assumption and the claim an element of $M_{D,h}$ that is also in $A_{\leq_P}$. This means that there exits a $\leq_P$-chain y in $S_{D,h}$ whose $\max_P$ is in $A_{\leq_P}$ and hence the whole $\leq_P$-chain is in $A_{\leq_P}$. Therefore (10) fails.
- The set $M_{D,h}$ is finite. Each chain in $S_{D,h}$ contains only elements which are $\leq_P x$ for some $x \in M_{D,h}$. By stability for each $x \in M_{D,h}$ there are only finitely many elements y with $x \geq_P y$. Applying Π^0_1-CP to this yields that there are only finitely elements y with $\exists x \in M_{D,h}\, y \leq_P x$ and hence that $S_{D,h}$ is finite.

In total (10) is equivalent to

$$\exists x \Big(\quad \forall y \left(y \text{ is } \leq_P\text{-chain} \wedge \max_P(y) > x \rightarrow y \notin S_{D,h}\right) \\ \wedge \forall y \left(y \text{ is } \leq_P\text{-chain} \wedge \max_P(y) \leq x \rightarrow y \notin S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right)\Big)$$

where the second quantification over y can be bounded and hence (10) is Δ^0_2.

Therefore an instance of Δ^0_2-IA (which is provable from an instance of Π^0_1-CA, see [21, Lemma 12.(iii)]) is sufficient to prove that there exists a maximal $D' \subseteq [0, n]$ for which $S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ is not empty, i.e.

$$\exists D' \subseteq [0, n]\, \exists h' \left(h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right. \\ \left.\wedge \forall E \left(D' \subseteq E \subseteq [0, n] \rightarrow \forall h' \left(h' \notin S_{E,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right)\right)\right).$$

Since D' is maximal each $h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ satisfies (9).

Hence taking for $g_1(n, h)$ the function that chooses for h and n an $h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ for this maximal D' has the desired properties. This choice function exists by an instance of Σ^0_2-AC which is also provable from an instance of Π^0_1-CA.

Now define g_2 to be a function which extends each chain $h \subseteq^{fin} A_{\leq_P}$ by one element in $A_{\leq_P}$, for instance

$$g_2(h) := h \cup \left\{ \min\{x \in A_{\leq_P} \mid \max(h) < x \wedge \max_P(h \cap A_{\leq_P}) \leq_P x\}\right\}.$$

This function exists also by an instance of Σ^0_2-AC.

The function $f(h) := g_2(g_1(\max(h), h))$ now satisfies the properties in (8) on page 13. By the discussion in the previous section the set $H := \bigcup_n f^n(\emptyset)$ is definable in this system and provably unbounded. The values of f are finite $\leq_P$-chains that are included in $A_{\leq_P}$. Hence H defines an unbounded $\leq_P$-chain.

Furthermore, one can prove Π^0_1-CA$(\varphi\chi_P H)$: To decide whether

$$\forall k\, \varphi\chi_P Hnk = 0 \tag{11}$$

holds for an n take an element $x \in H$ with $x \geq n$. By the unboundedness this exists. In particular there exists a smallest m such that $x \in f^m(\emptyset)$. For this we have $f^m(\emptyset) = f\left((\tilde{f}_x)^x(\emptyset)\right)$. By the definition g_1 and (9) we have that (11) is true iff

$$\forall k\, \alpha_{\varphi\chi_P}(g_1(|f^m(\emptyset)|, f^m(\emptyset)), n, k) \leq 1.$$

(We assume here again that H is given as strictly increasing enumeration.) This is again by the definition of g_1 true iff

$$\forall k\, \alpha_{\varphi\chi_P}(f^{m+1}(\emptyset), n, k) \leq 1.$$

Which is the same as

$$\forall k\, \alpha_{\varphi\chi_P}\big(ff\big((\tilde{f}_x)^x(\emptyset)\big), n, k\big) \leq 1$$

and thus can be computed using the imposed instance of comprehension.

The different instances of Π^0_1-CA can be coded together into a term ξ, see [21, Remark 11] and for a reference [17]. This solves the proposition. □

Corollary 14. CAC *is proofwise low over* $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \oplus \mathrm{WKL}$*.*

Proof. Lemma 13 from [21] for $n = 0$ shows that one can iterate f_H in the results of Proposition 13 while retaining the instance of comprehension. With this one can define an strictly increasing enumeration of H and hence shows that SCAC is proofwise low over $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC}$.

The result follows from the fact that COH is proofwise low of $\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} \oplus \mathrm{WKL}$ ([21, Corollary 18]) and from noting that the proof

$$\mathrm{SCAC} + \mathrm{COH} \rightarrow \mathrm{CAC}$$

in [11, Proposition 3.7] can be carried out in $\mathrm{G}_\infty\mathrm{A}^\omega$ while retaining the proofwise low property. □

Theorem 15. *The system*

$$\widehat{\text{WE-PA}}^\omega{\restriction} + \text{QF-AC} \oplus \mathrm{WKL} \oplus \mathrm{CAC}$$

is conservative over $\widehat{\text{WE-HA}}^\omega{\restriction}$ *for sentences of the form* $\forall x^1\, \exists y^0\, A_{qf}(x, y)$*. Moreover one can extract a primitive recursive realizing term* $t[x]$ *for* y*.*

In particular,

$$\mathrm{WKL}_0^\omega + \mathrm{CAC}$$

is conservative for sentences of the from $\forall x^1\, \exists y^0\, A_{qf}(x, y)$ *and a fortiori* Π^0_3*-conservative over* RCA_0^ω*.*

Proof. Corollary 14 and Corollaries 10, 11. □

This result raises the question whether one can extend it and show that RT^2_2 is proofwise low over a system like $\mathrm{WKL}_0^{\omega *}$ or any other system without Σ^0_1-induction and thus can show that RT^2_2 does not imply Σ^0_2-induction.

Let the Erdős-Moser principle (EM) be the principle that states that every tournament on $\mathbb{N}$ contains an infinite transitive subgraph. A tournament is a directed graph $\langle \mathbb{N}, \rightarrow \rangle$ such that for each pairs of nodes x, y either $x \rightarrow y$ or $x \leftarrow y$. The principle RT^2_2 is equivalent to $\mathrm{CAC} + \mathrm{EM}$ (in fact even to $\mathrm{ADS} + \mathrm{EM}$), see Appendix A. Corollary 14 shows that is sufficient to show the EM is proofwise low over a system without Σ^0_1-induction in order to show that RT^2_2 does not imply Σ^0_2-induction.

Appendix A. The Erdős-Moser principle

A tournament is a directed graph $\langle E, \to \rangle$ such that for each pairs of nodes x, y with $x \neq y$ either $x \to y$ or $x \leftarrow y$ but not both. The Erdős-Moser principle (EM) states that each tournament on $\mathbb{N}$ contains an infinite transitive subtournament. It is easy to see that EM follows from RT^2_2 if one identifies the tournament with the following 2-coloring of pairs of $\mathbb{N}$: For $x < y$ let

$$\begin{aligned} c(\{x,y\}) &= 0 \quad \text{iff} \quad x \to y, \\ c(\{x,y\}) &= 1 \quad \text{iff} \quad x \leftarrow y. \end{aligned} \tag{12}$$

On any homogeneous set of c the relation $\to$ is transitive. Hence RT^2_2 yields an infinite transitive subtournament.

In the other direction EM and ADS (the principle CAC restricted to linear orderings) imply RT^2_2. To see this let for some coloring c the relation $\to$ be defined by (12). Using EM one finds an infinte subset on which $\to$ is a linear ordering. The principle ADS yields an infinite $\to$-chain. By definition c is constant on this chain.

The principle EM was introduced by Bovykin and Weiermann in [3]. They also proved the above stated equivalence.

We now give some lower bounds on the strength of EM:

Proposition 16.

$$\mathrm{RCA}_0 \vdash \mathrm{EM} \to \Pi^0_1\text{-CP}$$

Proof. We show that EM proves the infinite pigeonhole principle. The result follows from this by [12].

Let $f \colon \mathbb{N} \to n$ be coloring of $\mathbb{N}$ with n colors. We consider the following infinite tournament. For $x < y$ let

$$\begin{aligned} x \to y \quad &\text{iff} \quad f(x) = f(y), \\ x \leftarrow y \quad &\text{iff} \quad f(x) \neq f(y). \end{aligned}$$

Applying EM yields and an infinite set X on which $\to$ is transitive. We claim that f restricted to X eventually becomes constant. Suppose not, then

$$\forall k \in X \, \exists x \in X \; (k < x \wedge f(k) \neq f(x))$$

which is by definition of $\to$

$$\forall k \in X \, \exists x \in X \; (k < x \wedge k \leftarrow x)$$

Now applying Σ^0_1-induction we obtain $n+1$ elements $x_1, \dots, x_{n+1} \in X$ with

$$x_1 < x_2 < \dots < x_{n+1} \qquad \text{and} \qquad x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_{n+1}.$$

By transitivity and definition of $\to$ we obtain that $f(x_i)$ are pairwise different. But this contradicts the fact that f is bounded by n.

The infinite pigeonhole principle for f and hence the proposition follows from this. ☐

Proposition 17. *There exists a computable tournament $\langle \mathbb{N}, \to \rangle$ that has no low infinite transitive subtournament, i.e. no set X such that $\to$ is transitive on X and $X' \leq_T 0'$.*

Proof. By [9] there exists a computable stable 2-coloring of pairs c, such that there is no low homogeneous set. Let $\to$ be the corresponding tournament as described by (12).

Suppose now that there is a low set X on which $\to$ is transitive and hence a linear ordering. Since c is stable this ordering is also stable. By Theorem 2.11 of [11] there exists an infinite chain Y that is low relative to X and hence low. Since on this chain the coloring c is homogeneous, this contradict the choice of c. □

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