The cohesive principle and the Bolzano-Weierstraß principle

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The Bolzano-Weierstraß principle

A real number is a sequence of rational numbers with Cauchy-rate $2^{-n}$.

**Definition**

(BW):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ has a cluster point.

*Equivalently,* every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence $(y_n)_n$ with Cauchy-rate $2^{-n}$, i.e. with

$$\forall n \forall i, j \geq n \ (|y_i - y_j| < 2^{-n}).$$

**Definition**

(BW\text{\_weak}):
Every bounded sequence $(x_n)_n \subseteq \mathbb{R}$ contains a Cauchy-subsequence $(y_n)_n$, i.e.

$$\forall n \exists k \forall i, j \geq k \ (|y_i - y_j| < 2^{-n}).$$
Assume that $(x_n)_n \subseteq [0, 1] \cap \mathbb{Q}$.

Goal: Construct a subsequence $(y_n)$ with $\forall n \forall i, j \geq n \ (|y_i - y_j| < 2^{-n})$.

- bi-partition argument
- $y_n := \begin{cases} \text{next element of } (x_n) \\ \text{in the } n\text{-th partition} \\ \text{chosen} \end{cases}$

The partitions form a $\Pi^0_2$-$0/1$-tree.

This is a $\Pi^0_1$-$0/1$-tree in $0'$.

WKL relativized to $0'$ yields an infinite branch and therefore computes the sequence of partitions.
Theorem (Kohlenbach '98, Kohlenbach, Safarik '10, K. '10)

- For each computable sequence \((x_n)\) there is a \(0'\)-computable \(0/1\)-tree \(T\), such that an infinite branch of \(T\) computes a cluster point, and vice versa.

- Over \(\text{RCA}_0\) the principles \(\text{BW}\) and \(\text{WKL}\) for \(\Sigma^0_1\)-trees are instance-wise equivalent.

By the low basis theorem:

Corollary

\(\text{BW}\) has for computable instances a solution \(\text{low}\) relative to \(0'\), i.e. the first Turing jump of a solution is computable in \(0''\).
Assume that \((x_n)_n \subseteq [0, 1] \cap \mathbb{Q}\).

Goal: Construct a subsequence \((y_n)_n\) with \(\forall n \exists k \forall i, j \geq k \ (|y_i - y_j| < 2^{-n})\) and compute the Turing jump of \((y_n)_n\).

It is clear that

\[
\Phi_e (y_n)_n \downarrow \text{ iff } \exists k \Phi_e (y_n)^{<k} \downarrow.
\]

Suppose that \((y_n)_n^{<m}\) is an initial segment that has already been computed. Deciding, whether there is an extension \((y_n)_n^{<l}\), such that

\[
\Phi_e (y_n)_n^{<l} \downarrow
\]

can be done in \(0'\).
Assume that \((x_n)_n \subseteq [0, 1] \cap \mathbb{Q}\).

Goal: Construct a subsequence \((y_n)\) with \(\forall n \exists k \forall i, j \geq k \ (|y_i - y_j| < 2^{-n})\) and compute the Turing jump of \((y_n)\).

- bi-partition argument
- now add at each step not only one element but finitely many elements of the chosen interval to \((y_n)\).

Let \((y_n)_{n<m}\) be the initial segment of \((y_n)\) computed up to the \(k\)-th step.

At the \(k\)-th step extend this to \((y_n)_{n<l}\) by elements in the \(k\)-th chosen interval, such that \(\Phi_k^{(y_n)_{n<l}} \downarrow\), if possible.

Then extend this by another element of the interval.
Theorem (K.)

For each bounded, computable sequence \((x_n)\)
there is a Cauchy-subsequence \((y_n)\),
such that \((y_n)\) and \((y_n)'\) are computable in
a Turing degree that contains infinite branches of \(0'\)-computable 0/1-trees.

Corollary

\(\text{BW}_{\text{weak}}\) has low\(_2\) solutions,
i.e. \((y_n)''\) is computable in \(0''\).

Proof.

\[(y_n)' \leq_T 0' + \text{WKL} \implies (y_n)'' \leq_T 0''\]

\(\text{BW}_{\text{weak}}\) does not compute \(0'\)
and is therefore strictly weaker than \(\text{BW}\).
The cohesive principle

Write \( X \subseteq^* Y \) if \( X \setminus Y \) is finite.

**Definition**

- A set \( X \) is *cohesive* for a sequence of set \((R_n)_n \subseteq 2^\mathbb{N}\) if
  \[
  X \subseteq^* R_n \lor X \subseteq^* \overline{R_n}
  \]
  for each \( n \).

- The *cohesive principle* (COH) states that for each \((R_n)_n\) there is an infinite cohesive set \( X \).

**Theorem (K.)**

- For each sequence \((x_n)_n \subseteq \mathbb{R}\) there exists \((R_n)_n \subseteq 2^\mathbb{N}\), such that from an infinite cohesive set for \((R_n)_n\) one can compute a Cauchy-subsequence of \((x_n)_n\) and vice versa.

- \( \text{RCA}_0 \vdash \text{BW}_{\text{weak}} \iff \text{COH} \land B\Sigma^0_2 \)

Moreover, this equivalence also holds instance-wise.
Theorem

- COH and hence also $BW_{\text{weak}}$ do not compute solutions to WKL in general. (Cholak, Jockusch, Slaman ’01)
- There are instances of these principles which have no low solutions. (Jockusch, Stephan ’93)

Proof of the $\text{low}_2$-ness of $BW_{\text{weak}}$ is a streamlined version of the $\text{low}_2$-ness of COH (Jockusch, Stephan ’93).
Theorem (Chong, Slaman, Yang ’10)

\[ \text{RCA}_0 + \text{COH} + B\Sigma^0_2 \text{ and thus } \text{RCA}_0 + \text{BW}_{\text{weak}} \]
are \( \Pi^1_1 \)-conservative over \( \text{RCA}_0 + B\Sigma^0_2 \).

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Theorem (K., Kohlenbach ’10)

If \( \text{WKL}_0 + \text{BW}_{\text{weak}} \vdash \forall f \exists y \phi(f, y) \)
for quantifier free \( \phi \),
then one can extract from a given proof
a **primitive recursive** function(al) \( t \) such that \( \forall f \phi(f, t(f)) \).

“Proof mining”
Bolzano-Weierstraß in the weak topology

We consider the Hilbert space $\ell_2 = (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$.
An element of $\ell_2$ is given by a Cauchy-sequence $(w_n)_n$ of finite dimensional and rational approximations, i.e. $w_n \in \mathbb{Q}^{<N}$, with Cauchy-rate $2^{-n}$ with respect to $\| \cdot \|$.

**Definition**

(weak-BW): Every $\| \cdot \|$-bounded sequence $(x_n) \subseteq \ell_2$ has a weak cluster point $x$, i.e. $\forall y \in \ell_2 \lim_{n \to \infty} \langle y, x_n \rangle = \langle y, x \rangle$.

**Theorem (K.)**

- For each bounded sequence $(x_n) \subseteq \ell_2$ there is a weak cluster point $x$ computable in $0''$.
- There is a bounded and computable sequence $(x_n) \subseteq \ell_2$, such that each weak cluster point of it computes $0''$.
- Over RCA$_0$ the principles $\Pi^0_2$-CA and weak-BW are instance-wise equivalent.
• $BW$ is equivalent to $WKL$ for $0'$-computable trees.
• $BW_{\text{weak}}$ is equivalent to $COH$.
  • Hence, it does not imply $0'$.
  • It admits extraction of primitive recursive terms.
• $\text{weak-BW}$ is equivalent to $0''$. 
References

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